Karlsruhe Institute of Technology

Bachelor's Thesis

Construction of Thom Spectra for Bordism with G-Structure

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Abstract

The Pontryagin-Thom construction states that framed bordism classes of framed submanifolds of a certain smooth manifold are in bijection with smooth homotopy classes of smooth maps of that manifold to a sphere. We give the proof of this theorem in the first part of this thesis. The second and third parts generalise this result to yield connections to homotopy theory and homology theory. First we omit the reference to an embedding into a specific manifold through a stabilisation process to obtain a correspondence to stable homotopy groups. Finally, we allow not only framings but more general structures on our manifolds and bordisms leading to general homology theories defined in terms of spectra.

Zusammenfassung

Die Pontryagin-Thom Konstruktion liefert eine Bijektion zwischen gerahmten Bordismusklassen gerahmter Untermannigfaltigkeiten einer gegebenen Mannigfaltigkeit und Homotopieklassen von Abbildungen dieser Mannigfaltigkeit in eine Sphäre. Im ersten Teil dieser Arbeit beweisen wir diesen Satz. Im zweiten und dritten Teil verallgemeinern wir dieses Ergebnis dann, um Verbindungen zur Homotopie- und Homologietheorie herzustellen. Zunächst lösen wir uns durch einen Stabilisierungsprozess von Untermannigfaltigkeiten und können Mannigfaltigkeiten unabhängig von einer Einbettung betrachten. An dieser Stelle finden wir stabile Homotopiegruppen in unserer Theorie wieder. Zuletzt verallgemeinern wir die zu Beginn betrachteten Rahmungen und erlauben allgemeinere Strukturen auf unseren Mannigfaltigkeiten und Bordismen. Wir beweisen einen Isomorphismus der Bordismusgruppen zu verallgemeinerten Homologiegruppen.

Contents

1. Introduction

Motivation

The general idea of bordism theory is to consider manifolds up to boundary, i.e. to consider manifolds up to the equivalence relation called *bordism* generated by two manifolds N_1 and N_2 being bordant if they are the common boundary of a higher-dimensional manifold M. While this is easy to understand and visualise, it does not yield a very interesting theory yet: Regarding the disjoint union $\coprod$ as a sum operation, we see that with respect to this structure, every element is of order two. Taking the disjoint union $M \coprod M$ of any manifold with itself, we obtain a *null-bordant* manifold (a manifold bordant to the empty manifold) because $\partial(M \times [0,1]) = M \coprod M$. This algebraic structure is not diverse enough to answer interesting questions in topology. So we need to enrich our theory by restricting the bordisms we allow in order to obtain a more general theory.

We first take a very direct approach at defining the structure of a *framing* on a submanifold and require bordisms to admit a framing compatible with that of the boundary. This yields a remarkable result connecting differential topology and homotopy theory.

While this result for framed bordisms is already very valuable by itself, it also motivates considering other structures on submanifolds and bordisms than just framings. These more general structures are defined in terms of the normal bundle of a submanifold of a sphere. Although the use of normal bundles requires us to consider submanifolds of certain manifolds, it is possible to *stabilise* these normal bundles to obtain a stable structure on a manifold independent of any embedding. The main aim of this thesis is to prove *Thom's Theorem*, which states an isomorphism between bordism classes with respect to some additional structure and generalised homology groups defined via spectra called *Thom Spectra* arising from the stable structure on a manifold.

By *manifold* we shall mean an m-dimensional compact and smooth manifold with or without boundary. A *submanifold* shall be an embedded submanifold.

Source material

The first part of my thesis is based on *Topology from the differentiable viewpoint* by John W. Milnor [8] and a seminar taking place in the summer semester 2017 at KIT. In the second part of my thesis I used Davis and Kirk's *Lecture notes in algebraic topology* [3] as well as chapter 3.3 from Lück's *Basic Introduction to Surgery Theory* [7] to get an overview of the topic and the necessary background knowledge, and then worked with a wide rage of sources, most prominently Bröcker and tom Dieck's *Kobordismentheorie* [1].

Acknowledgement

I want to thank my advisor Holger Kammeyer for making it possible for me to write my Bachelor's thesis without being in Karlsruhe most of the time and still being able to advise me whenever I needed advice. It made me work very intensively by myself and strengthen my own initiative and endurance. Whenever needed, I could always contact him with questions or uncertainties.

2. Framed Bordism and the Pontryagin-Thom Construction

In this first chapter we make several simplifications to understand the general concept of bordism. We do not yet consider stable bordism but instead consider bordism within a specified manifold. So we only consider bordisms between submanifolds N_1^n, N_2^n of M^m of a specific dimension. While we will consider manifolds mapping to a specific manifold X later, called *manifolds in* X, in order to compute generalised homology groups of several manifolds, we leave out this additional condition throughout this chapter. By setting $X = \mathrm{pt}$ this condition becomes trivial since every manifold has a unique mapping to a point. The structure we require on our submanifolds N^n of M^m and bordisms is a *framing*. A framing is a basis of the normal space of N^n in M^m at each point of N, varying continuously. In terms of the normal bundle this is a trivialisation of the normal bundle or equivalently a reduction of the structure group of the normal bundle to the trivial group, as will be explained in more detail later. So for some m-dimensional manifold and some n we will investigate

$$\Omega^1_{n,M}(\mathrm{pt}) =: \Omega^{fr}_{n,M},$$

the framed bordism classes of n-dimensional framed submanifolds of M^m.

The focus of this chapter will lie on proving the one-to-one correspondence between smooth homotopy classes of smooth maps $f\colon M \to S^{m-n}$, where M is an m-dimensional compact, boundaryless manifold, and framed bordism classes of submanifolds of codimension $p := m - n$ in M. The precise terminology will be introduced later. However, one can easily see that once one has understood the framed bordism classes of submanifolds of codimension p of S^m, one knows the smooth homotopy classes of smooth maps $f\colon S^m \to S^p$ and thus is very close to understanding $\pi_m(S^p)$, a very hard problem. For specific m and p, this will be our first application of the theory.

$$\boxed{\text{Aim: } \Omega^{fr}_{n,M^m} \overset{1:1}{\longleftrightarrow} [M^m, S^{m-n}]}$$

This chapter uses [8] as its main source. All definitions, theorems and lemmata can be found in chapter 7 of [8]. Only additional sources are mentioned explicitly.

2.1. The Bordism Relation

Let N_1^n and N_2^n be compact submanifolds of the manifold M^m with $\partial N_1^n = \partial N_2^n = \partial M^m = \emptyset$. The difference of dimensions $m-n$ is called *codimension* of N_1^n respectively N_2^n within M^m.

In this chapter all manifolds are considered submanifolds of $\mathbb{R}^k$ for some $k \in \mathbb{N}$. By the Whitney Embedding Theorem [A.1] this does not restrict the set of manifolds considered. These embeddings into some $\mathbb{R}^k$ yield a metric on each manifold and a scalar product on each tangent space.

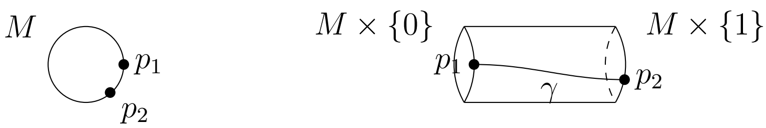
Definition 2.1. N_1^n is *bordant* to N_2^n within M^m if for some $\epsilon > 0$ the subset $N_1 \times [0, \epsilon) \cup N_2 \times (1-\epsilon, 1]$ of $M \times [0,1]$ can be extended to a compact manifold $X \subset M \times [0,1]$ such that $\partial X = N_1 \times \{0\} \cup N_2 \times \{1\}$ and so that $X \cap (M \times \{0\} \cup M \times \{1\}) = \partial X$.

Remark.

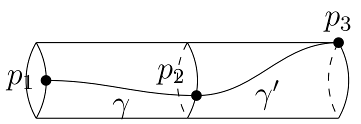
1. For X to have as boundary the n-dimensional manifold $N_1 \times \{0\} \cup N_2 \times \{1\}$, X needs to have dimension $n+1$.
2. The right inclusion $X \cap (M \times \{0\} \cup M \times \{1\}) \supset \partial X$ follows from $\partial X = N_1 \times \{0\} \cup N_2 \times \{1\} \subset (M \times \{0\} \cup M \times \{1\})$, whereas the left inclusion $X \cap (M \times \{0\} \cup M \times \{1\}) \subset \partial X$ requires that X does not intersect $M \times \{0\} \cup M \times \{1\}$ except at the points of ∂X, i.e. at N_1 respectively N_2.
3. Bordism is an equivalence relation. For this we need that $N_1 \times [0, \epsilon) \subset M \times [0,1]$ and $N_2 \times (1-\epsilon, 1] \subset M \times [0,1]$ are extended to a bordism $X \subset M \times [0,1]$ and not only $N_1 \times \{0\}$ and $N_2 \times \{1\}$. This ensures that two bordisms glued together yield again a smooth manifold as indicated in the picture following Example 2.4.

Example 2.2. The simplest example of a bordism is $N_1 := M \times \{0\}$, $N_2 := M \times \{1\} \subset M \times [0,1]$ with $X = M \times [0,1]$ for any compact manifold without boundary M.

Example 2.3. Consider $M = S^1$ as the unit circle in $\mathbb{R}^2$ and $N_1 = \{p_1\}, N_2 = \{p_2\}$ as two points on M. Then $M \times [0,1]$ is a cylinder of radius 1 and length 1. The manifold X can be taken as the image of a path $\gamma\colon [0,1] \to M \times [0,1]$ from $N_1 \subseteq M \times \{0\}$ to $N_2 \subset M \times \{1\}$ satisfying $\gamma(t) = (p_1, t)$ for $0 \leq t < \epsilon$ and $\gamma(t) = (p_2, t)$ for $1 - \epsilon < t \leq 1$ as indicated in the picture below:



Example 2.4. Considering once more $M = S^1$ with now three points $N_1 = \{p_1\}, N_2 = \{p_2\}, N_3 = \{p_3\}$ on the sphere, we can glue together two bordisms from N_1 to N_2 and from N_2 to N_3 as in the following picture:



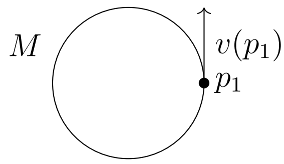
2.2. Framings

Definition 2.5. A *framing* of a submanifold $N^n \subset M^m$ with codimension $p := m - n$ is a smooth function $v\colon N \to ((TN)^\perp)^p$ which assigns to each $x \in N$ a basis

$$v(x) = (v_1(x), \dots, v_p(x))$$

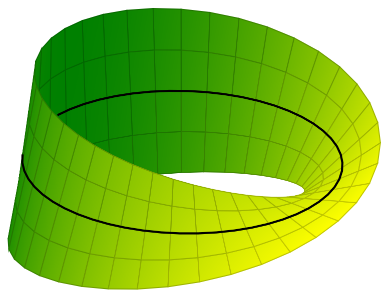
of $(T_xN)^\perp \subset T_xM$ – the space of normal vectors to N in M at x.

The pair (N, v) is called *a framed submanifold* of M.



Remark. $(TN)^\perp$ is well-defined, because we are considering $N \subset M \subset \mathbb{R}^k$.

Example 2.6. Not all submanifolds are framable: For example the 1-sphere $S^1 \subset M$ as a submanifold of the Möbius band M, embedded as indicated in the picture below [5] cannot be framed. A function $v\colon S^1 \to ((TS^1)^\perp)$ such that $v(x)$ is a basis of $(T_xS^1)^\perp$ for each $x \in S^1$ cannot be continous, as "walking around" S^1 once "flips" the orientation.



Remark. Note that for manifolds $N' \subset M'$ of dimensions n and m with boundary, even for a boundary point $x \in N'$, T_xN' is an n-dimensional vector space and T_xM' an m-dimensional vector space [8, Ch. 2]. So a framing of a manifold with boundary can be defined in exactly the same way as done above for a manifold without boundary.

Definition 2.7. Two n-dimensional framed submanifolds (N_1, v) and (N_2, w) are called *framed bordant* within M^m if there exists a framing $u\colon X \to ((TX)^\perp)^{m-n}$ of a bordism $X \subset M \times [0, 1]$, such that

$$u_i(x, t) = (v_i(x), 0) \text{ for } (x, t) \in N_1 \times [0, \epsilon) \subset X,$$

and

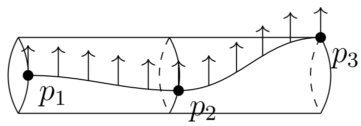
$$u_i(x, t) = (w_i(x), 0) \text{ for } (x, t) \in N_2 \times (1 - \epsilon, 1] \subset X.$$

We denote the set of framed bordism classes of n-dimensional framed submanifolds of M^m by $\Omega^{fr}_{n,M}$.

Remark. 1. We require our framed bordism to be constant in an ϵ-interval around the boundary to ensure that the composition of two framings is again a smooth function.

2. Framed bordism is an equivalence relation. The only issue is the smoothness of the framing when concatenating two framed bordisms to show transitivity. This however is ensured by the resulting framing being constant in an ϵ-neighbourhood around the junction.

Example 2.8.



2.3. The Pontryagin Manifold

Note. There is a naturally induced orientation on S^p: The standard orientation on $\mathbb{R}^{p+1}$ induces an orientation on the closed unit $(p+1)$-ball $D^{p+1} \subset \mathbb{R}^{p+1}$. Then we can equip S^p with the boundary orientation coming from the closed unit ball.

Consider any smooth map $f\colon M \to S^p$ from a compact m-dimensional manifold to the p-sphere, where $p \leq m$. Let $y \in S^p$ be a regular value. Note that for $p > m$ regular values do not exist because $D_x f\colon \mathbb{R}^m \cong T_xM \to T_xS^p \cong \mathbb{R}^p$ cannot be surjective. For $p \leq m$ such a regular value does exist because by Brown's Corollary found in [8, Ch. 2], the set of regular values of $f\colon M \to S^p$ is everywhere dense in S^p.

By the regular value theorem (or preimage theorem), $X := f^{-1}(y)$ is an $(m-p)$-dimensional submanifold of M. We obtain a framing of X through the following procedure:
Choose a positively oriented basis $v = (v_1, ..., v_p)$ for the tangent space T_yS^p. By Lemma 2 in [8, Ch. 2], $\ker(D_xf\colon T_xM \to T_yS^p) = T_xX$ for each $x \in X$. Thus, $D_xf_{|(T_xX)^\perp}\colon (T_xX)^\perp \xrightarrow{\cong} T_yS^p$ is an isomorphism. So for each $x \in X$ and each $i \in \{1, ..., p\}$, there exists exactly one $w_i(x) \in (T_xX)^\perp$ that maps to $v_i(x)$ under D_xf. Then $w = (w_1(x), \dots, w_p(x)) =: f^*v$ is a framing of $X = f^{-1}(y)$. Continuity and triviality at the boundaries are easy to check.

This resulting framed manifold $(f^{-1}(y), f^*v)$ will be called the *Pontryagin manifold* associated with f.

Remark. Now consider any smooth map $F\colon M' \to S^p$ where M' is an $(m+1)$-dimensional manifold with boundary and $p \leq m$. Let $y \in S^p$ be a regular value for both F and $F_{|\partial M'}$. Then by [8, Ch. 2, Lemma 4], $F^{-1}(y)$ is a smooth $(m-p+1)$-dimensional manifold with $\partial(F^{-1}(y)) = F^{-1}(y) \cap \partial M'$. By [8, Ch. 2, Lemma 2] the above works just as well for manifolds with boundary. Hence the Pontryagin manifold can also be constructed in this setting. This will be important in the proof of two upcoming lemmata where we want to use the Pontryagin manifold associated with a homotopy $F\colon M \times [0,1] \to S^p$.

Since we made several choices (we chose a regular value and a positively oriented basis), "the Pontryagin manifold" is not yet well-defined. For this definition of "the Pontryagin manifold" to be valid, we need to show that different choices of y and v above lead to the "same" manifold. The classification we want to accomplish here is up to framed bordism. We will show that all manifolds $(f^{-1}(y), f^*v)$ corresponding to different choices of y and v belong to a single framed bordism class. Once we have shown that, we can accept the following definition:

Definition 2.9. The framed bordism class of $(f^{-1}(y), f^*v)$ is called the *Pontryagin manifold* associated with f.

The following theorem states what we need:

Theorem 2.10. *If y' is another regular value of f and v' is any positively oriented basis for $T_{y'}S^p$, then the framed manifold $(f^{-1}(y), f^*v)$ is framed bordant to $(f^{-1}(y'), f^*v')$.*

To simplify the proof of the above theorem, we split it into three lemmata.

Lemma 2.11. *If v and v' are two different positively oriented bases of T_yS^p, then the Pontryagin manifold $(f^{-1}(y), f^*v)$ is framed bordant to $(f^{-1}(y), f^*v')$.*

Proof. The positively oriented bases of T_yS^p are precisely those that can be reached from a positively oriented basis – such as v – through multiplication with a transformation matrix of positive determinant. Thus we can identify the space of positively oriented bases with the space of real matrices with positive determinant: $GL_p^+(\mathbb{R})$. This space is path-connected [2] and thus so is the space of positively oriented bases. Hence we can choose a smooth path u from v to v'. We can adjust this path at the end points such that $u_{|[0,\epsilon)} = v$ and $u_{|(1-\epsilon,0]} = v'$. Then $(f^{-1}(y) \times [0,1], u)$, meaning $f^{-1}(y) \times t$ with framing $u(t)$ for $t \in [0,1]$, is a framed bordism for $(f^{-1}(y), f^*v)$ and $(f^{-1}(y), f^*v')$. □

Since the choice of framing v does not change the framed bordism class, we will often omit the reference f^*v and only speak of the framed manifold $f^{-1}(y)$.

Lemma 2.12. *If y is a regular value of $f\colon M \to S^p$ and z is sufficiently close to y, then $f^{-1}(z)$ is framed bordant to $f^{-1}(y)$.*

Proof. Since M and S^p are compact manifolds, we can choose finite atlases $(\{U_i, \varphi_i\})$ and $(\{V_j, \psi_j\})$. We can now express f locally as $f_{ij}\colon \mathbb{R}^m \supset \varphi_i(U_i) \to \psi_j(V_j) \subset \mathbb{R}^p$ with differential $Df_{|TU_i} = Df_{ij}$ given by the Jacobian of f_{ij}. The critical points of f are those $x \in U_i$ where $\operatorname{rank}(Df_{ij}(x)) < p$. This set of critical points in a chart U_i is closed: For any regular point $x \in U_i$, the rows of $Df_{ij}(x)$ are linearly independent. Since the entries of $Df_{ij}(x)$ vary smoothly with $x \in U_i$, there exists an open neighbourhood $U_x \subset U_i$ of x such that the rows of $Df_{ij}(y)$ are still linearly independent for $y \in U_x$ and $Df_{ij}(y)$ thus also has full rank. Hence the set of regular points in U_i is open and its complement, the set of critical points in U_i, is closed. Since there are only finitely many charts, the set of all critical points C in M is also closed and as a closed subset of a compact manifold it is compact. Since f is continuous, the set $f(C)$ of critical values in S^p is also compact, thus closed. So we can choose an $\epsilon > 0$ such that the ϵ-neighbourhood of y contains only regular values. We now show that for any z in this ϵ-neighbourhood the lemma holds.

Let $z \in S^p$ with $||z - y|| < \epsilon$. Choose an isotopy $r\colon S^p \times [0,1] \to S^p$ such that

1. $r_1(y) = z$,
2. $r_t = \mathrm{id}_{S^p}$ for $t \in [0, \epsilon')$, for some $\epsilon' > 0$,
3. $r_t = r_1$ for $t \in (1-\epsilon', 1]$,
4. each $r_t^{-1}(z)$ lies on a great circle from y to z – i.e. a shortest path from y to z – therefore has distance less than ϵ to y and is thus a regular value of f.

This isotopy can be chosen as a family of rotations along a great arc from y to z.

Now define the homotopy $F\colon M \times [0,1] \to S^p$ between f and $r_1 \circ f$ as

$$F(x,t) := r_t(f(x)).$$

For each $t \in [0,1]$, z is a regular value of r_t. By the requirements above $r_t^{-1}(z)$ is a regular value of f, so z is a regular value of $r_t \circ f$. For z to be a regular value of F, we need each $(x,t) \in F^{-1}(z)$ to be a regular point of F. Since $F_{|\{(x,t)\}} = (r_t \circ f)_{|\{x\}}$ and $x \in (r_t \circ f)^{-1}(z)$, this is the case. Thus, z is a regular value of F. Following the procedure above, $F^{-1}(z) \subset M$ is a framed manifold providing a framed bordism between the framed submanifolds $(r_0 \circ f)^{-1}(z) = f^{-1}(z) \subset M \times \{0\}$ and $(r_1 \circ f)^{-1}(z) = f^{-1}(r_1^{-1}(z)) = f^{-1}(y) \subset M \times \{1\}$. □

Lemma 2.13. *If $f\colon M \to S^p$ is smoothly homotopic to $g\colon M \to S^p$ and y is a regular value for both, then $f^{-1}(y)$ is framed bordant to $g^{-1}(y)$.*

Proof. Since f and g are smoothly homotopic, there is a smooth homotopy $H\colon M \times [0,1] \to S^p$ from f to g. By walking through the homotopy a little faster we obtain a smooth homotopy $F\colon M \times [0,1] \to S^p$ with

$$F(x,t) = f(x) \text{ for } 0 \le t < \epsilon,$$

and

$$F(x,t) = g(x) \text{ for } 1 - \epsilon < t \le 1.$$

As seen in the proof of the previous lemma there is an open neighborhood $U \subset S^p$ of y that only contains regular values of f and g. Since the regular values of F are dense in S^p by the Theorem of Sard [A.5], we can choose a regular value $z \in U$ of F. Then $f^{-1}(y)$ is framed bordant to $f^{-1}(z)$ and $g^{-1}(y)$ is framed bordant to $g^{-1}(z)$ by the previous lemma. So $F^{-1}(z) \subset M \times [0,1]$ is a framed manifold (by the Pontryagin construction) and provides a framed bordism between $f^{-1}(z)$ and $g^{-1}(z)$: By choice of the homotopy, $F(f^{-1}(z),t) = z$ for $0 \le t < \epsilon$ and $F(g^{-1}(z),t) = z$ for $1-\epsilon < t \le 1$, so $f^{-1}(z) \times [0,\epsilon) \cup g^{-1}(z) \times (1-\epsilon, 1] \subset F^{-1}(z)$. As the preimage of a closed set and subset of a compact manifold, $F^{-1}(z)$ is compact and $\partial F^{-1}(z) = F^{-1}(z) \cap (M \times \{0\} \cup M \times \{1\}) = f^{-1}(z) \times \{0\} \cup g^{-1}(z) \times \{1\}$ [4, Prop. 2.22]. $D_{(x,t)}F = D_x f$ for $0 \le t < \epsilon$ and $D_{(x,t)}F = D_x g$ for $1 - \epsilon < t \le 1$, so the framing is also constant at the boundary.

Since framed bordism is an equivalence relation, $f^{-1}(y)$ is framed bordant to $g^{-1}(y)$ by transitivity applied twice. □

Proof of Theorem 2.10. Now given any two regular values y and z of $f\colon M \to S^p$ and any positively oriented bases v for T_yS^p and w for T_zS^p, we want to show that $(f^{-1}(y), v)$ is framed bordant to $(f^{-1}(z), w)$. By Lemma 2.11 we do not need to consider v and w.

Choose a smooth one-parameter family of rotations $r_t\colon S^p \to S^p$ such that r_0 is the identity and $r_1(y) = z$. For example the rotation of S^p along a great arc through y and z. This family of rotations is a smooth homotopy between $r_0 = \mathrm{id}$ and r_1 and hence $r_t \circ f$ is a smooth homotopy between f and $r_1 \circ f$. z is a regular value for both f and $r_1 \circ f$. By Lemma 2.13, $f^{-1}(z)$ is framed bordant to $(r_1 \circ f)^{-1}(z) = f^{-1}(r_1^{-1}(z)) = f^{-1}(y)$. □

2.4. The Pontryagin-Thom Isomorphism

Theorem 2.14 (Product Neighbourhood Theorem)**.** *Let $N^n \subset M^m$ be a framed submanifold of the manifold M with codimension p and framing v. Assume N to be compact with $\partial N = \partial M = \emptyset$.*

Then there is a neighbourhood $V \subset M$ of N in M diffeomorphic to $N \times \mathbb{R}^p$ via a diffeomorphism $\phi\colon N \times \mathbb{R}^p \xrightarrow{\cong} V$ so that

$$\phi(x, 0) = x$$

for all $x \in N$ and so that the standard basis of $\mathbb{R}^p$ corresponds to the normal frame $v(x) \subset (T_x N)^\perp$ under each $D_x\phi$, $x \in N$.

Conversely, every diffeomorphism $\phi\colon N \times \mathbb{R}^p \to V$ with $N \subset V \subset M$, N a smooth submanifold of M, yields a framing v of N in M.

Remark. The framing of the submanifold N is a necessary requirement, since product neighbourhoods do not exist for arbitrary submanifolds. For example there is no neighbourhood of $S^1 \subset M$ diffeomorphic to $S^1 \times \mathbb{R}^p$. Here $S^1 \subset M$ is the 1-sphere embedded into the Möbius band M as in Example 2.6.

Proof. "$\Rightarrow$": We first consider the simplified case of $M = \mathbb{R}^{n+p}$ and then the general case of M being any $(n+p)$-dimensional boundaryless submanifold of some $\mathbb{R}^k$.

Case 1: $M = \mathbb{R}^{n+p}$.
Consider the map

$$\phi\colon N^n \times \mathbb{R}^p \to \mathbb{R}^{n+p},\ (x; t_1, \dots, t_p) \mapsto x + t_1 v_1(x) + \cdots + t_p v_p(x).$$

ϕ clearly satisfies $\phi(x, 0) = x$ for all $x \in N$. Furthermore ϕ has differential

$$\mathbf{D}_{(\mathbf{x};\mathbf{0},\dots,\mathbf{0})}\phi = \begin{pmatrix} 1 & & 0 & & & & \\ & \ddots & & & & & \\ 0 & & 1 & & & & \\ & & & v_1(x)_1 & v_2(x)_1 & \dots & v_p(x)_1 \\ & & & v_1(x)_2 & v_2(x)_2 & \dots & v_p(x)_2 \\ & & & \vdots & \vdots & \ddots & \vdots \\ & & & v_1(x)_p & v_2(x)_p & \dots & v_p(x)_p \end{pmatrix}$$
$$= \begin{pmatrix} 1 & & 0 & & & & \\ & \ddots & & & & & \\ 0 & & 1 & & & & \\ & & & & & & \\ & & & (v_1(x)) & (v_2(x)) & \dots & (v_p(x)) \\ & & & & & & \end{pmatrix},$$

which is invertible for every $(x; 0, \dots, 0) \in N \times \{0\}$. By the implicit function theorem, there exists a neighbourhood $A \subset N \times \mathbb{R}^p$ of $(x; 0, \dots, 0)$ such that $\phi\colon A \to \phi(A)$ is a diffeomorphism, where $\phi(A)$ is an open subset of $\mathbb{R}^{n+p}$. Our aim is now to show that ϕ is a diffeomorphism in some neighbourhood $N \times B_\epsilon(0) \subset N \times \mathbb{R}^p$ of $N \times \{0\}$.

We proceed by showing that $\phi_{|N \times B_\epsilon(0)}$ is injective for some $\epsilon > 0$ by assuming the contrary, i.e. that there exist $(x, u) \neq (x', u') \in N \times \mathbb{R}^p$ for arbitrarily small $||u||$ and $||u'||$

with $\phi(x,u) = \phi(x',u')$. We can choose two sequences $(x_i,u_i)_{i\in\mathbb{N}}$, $(x'_i,u'_i)_{i\in\mathbb{N}}$ such that $x_i \xrightarrow{i\to\infty} x_0$, $u_i \xrightarrow{i\to\infty} 0$, $x'_i \xrightarrow{i\to\infty} x'_0$, $u'_i \xrightarrow{i\to\infty} 0$ and $\phi(x_i,u_i) = \phi(x'_i,u'_i)$ for all $i\in\mathbb{N}$. Since N is compact and this is a closed condition on the sequences, we get that $x_0, x'_0 \in N$ and $x_0 = \phi(x_0,0) = \phi(x'_0,0) = x'_0$. This is a contradiction to ϕ being a diffeomorphism – so in particular one-to-one – in a neighborhood of $(x_0,0) = (x'_0,0)$. Thus ϕ is injective on $N\times B_{\epsilon'}(0)$ for some $\epsilon'>0$ and $\phi\colon N\times B_{\epsilon'}(0)\to\phi(N\times B_{\epsilon'}(0))\subset\mathbb{R}^{n+p}$ is bijective. Since ϕ is a diffeomorphism in a neighbourhood of each $(x;0,\dots,0)\in N\times\{0\}$ and bijective on $B_{\epsilon'}(0)$, there is a $U := B_\epsilon(0)$ with $0<\epsilon<\epsilon'$ such that $\phi\colon N\times U\to\phi(V) := V$ is a diffeomorphism.

Furthermore, one can easily check that

$$\varphi\colon U\to\mathbb{R}^p,\ u\mapsto\frac{u}{1-\frac{||u||^2}{\epsilon^2}}$$

is a diffeomorphism. Hence

$$\overline{\phi} := \phi\circ(\operatorname{id}\times\varphi^{-1})\colon N\times\mathbb{R}^p \xrightarrow{\cong} V$$

is a diffeomorphism satisfying

$$\overline{\phi}(x,0) = \phi(x,0) = x \text{ and}$$

$$D_{(x;0,\dots,0)}\phi\cdot e_{n+i} = v_i(x)\in(T_xN)^\perp$$

for $i\in\{1,\dots,p\}$. This proves the first case.

<u>Case 2:</u> Let M be any boundaryless $(n+p)$-dimensional submanifold of $\mathbb{R}^k$ for some $k\geq n+p$. The idea is to replace the straight lines $t_iv_i(x)$ we used in the case $M=\mathbb{R}^{n+p}$ by geodesics in M.

For $(x;t_1,\dots,t_p)\in N\times\mathbb{R}^p$ let

$$h(x;t_1,\dots,t_p) := t_1v_1(x)+\dots+t_pv_p(x) \text{ and } w(x,t) := \frac{h(x,t)}{||h(x,t)||}\in(T_xN)^\perp\subset T_xM.$$

Let c be the unique arc-length parametrised geodesic on M with starting point x and initial velocity $w(x,t)$ (note that $||w(x,t)||=1$). Let $c(||h(x,t)||)$ be the endpoint of the geodesic subsegment of c of length $||h(x,t)||$. Then there exists some ϵ''-neighbourhood $W := B_{\epsilon''}(0)$ such that

$$\phi\colon N\times W\to M,\ \phi(x;t_1,\dots,t_p) := c(||h(x,t)||)$$

is well-defined and smooth. For more details on geodesics see [6, Ch. 4]. From here we can proceed as before:

$$\mathbf{D_{(x;0,\dots,0)}}\phi = \begin{pmatrix} 1 & & 0 & & & \\ & \ddots & & & & \\ 0 & & 1 & & & \\ & & & (v_1(x)) & (v_2(x)) & \dots & (v_p(x)) \\ & & & & & & \end{pmatrix}$$

is invertible, so ϕ is a diffeomorphism $\phi\colon A \to \phi(A)$ for a neighbourhood $A \subset N \times \mathbb{R}^p$ of each $(x, 0) \in N \times \{0\}$. The same argument as above shows that ϕ is a diffeomorphism $\phi : N \times U \to V \subset M$ in a neighbourhood U of $N \times \{0\} \subset N \times \mathbb{R}^p$. Precomposing with $(\mathrm{id} \times \varphi^{-1})$ as above yields the desired diffeomorphism $N \times \mathbb{R}^p \xrightarrow{\cong} V$.

"$\Leftarrow$": This direction is a lot simpler. Given a diffeomorphism

$$\phi\colon N \times \mathbb{R}^p \to V \subset M \text{ satisfying } \phi(x, 0) = x \text{ for } x \in N,$$

the differential $D_{(x;0,\ldots,0)}\phi\colon T_{(x;0\ldots0)}(N \times \mathbb{R}^p) \to M$ given by

$$\mathbf{D}_{(\mathbf{x};\mathbf{0},\ldots,\mathbf{0})}\phi = \begin{pmatrix} 1 & & 0 & & & \\ & \ddots & & & & \\ 0 & & 1 & & & \\ & & & (v_1(x)) & (v_2(x)) & \ldots & (v_p(x)) \\ & & & & & & \end{pmatrix}$$

is an isomorphism for each $(x; 0, \ldots, 0) \in T_{(x;0\ldots0)}(N \times \mathbb{R}^p) \cong T_xN \times \mathbb{R}^p$. Since it is the identity restricted to the tangent space of N, it maps $\mathbb{R}^p$ isomorphically onto the orthogonal complement $(T_xN)^\perp \subset T_xM$. Thus

$$v\colon N \to ((TN)^\perp)^p,\ x \mapsto (v_1(x), \ldots, v_p(x))$$

with

$$v_i(x) := D_{(x;0,\ldots,0)}\phi \cdot e_{n+i}$$

defines a framing of N in M. □

The product neighbourhood theorem gives us a very useful equivalent characterisation of framed submanifolds $N \subset M$ as submanifolds equipped with a product neighbourhood $\phi\colon N \times \mathbb{R}^p \xrightarrow{\cong} V \subset M$.

The following theorem states the surjectivity of the Pontryagin-Thom isomorphism $[M, S^{m-n}] \to \Omega^{fr}_{n,M}$ and uses the product neighbourhood theorem to prove this.

Theorem 2.15. *Any compact framed submanifold (N, v) of codimension p in M occurs as a Pontryagin manifold for some smooth mapping $f\colon M \to S^p$.*

Proof. Let $N \subset M$ be a compact boundaryless framed submanifold of codimension p with framing v. Choose a product representation $\phi\colon N \times \mathbb{R}^p \to V \subset M$ for some neighboorhood V of N as in the previous theorem and let $\pi\colon V \xrightarrow{\mathrm{pr}_2 \circ \phi^{-1}} \mathbb{R}^p$, i.e. $\pi(\phi(x, y)) = y$.

For $x \in N$, $\pi(x) = \mathrm{pr}_2(x, 0) = 0$ and

$$D_x\pi \cdot v_i(x) = e_i.$$

So the Pontryagin manifold $N = \pi^{-1}(0)$ carries the same framing v as we started off with.

Now let $\varphi : \mathbb{R}^p \to S^p$ be a smooth map with $\varphi(x) = s_0$ for all x with $||x|| \geq 1$ that maps $B_1(0)$ diffeomorphically onto $S^p \setminus \{s_0\}$ by setting $\varphi(x) := h^{-1}(\frac{x}{\lambda(||x||^2)})$, where $h : S^p \setminus \{s_0\} \to \mathbb{R}^p$ is the stereographic projection and $\lambda : \mathbb{R} \to \mathbb{R}$ is a smooth monotone decreasing function with $\lambda(t) > 0$ for $t < 1$ and $\lambda(t) = 0$ for $t \geq 1$. Here s_0 is the basepoint of S^p. This allows us to define $f : M \to S^p$ by

$$f(x) = \begin{cases} \varphi(\pi(x)) & \text{for } x \in V, \\ s_0 & \text{for } x \notin V. \end{cases}$$

f is smooth and $\varphi(0)$ is a regular value of f. The associated Pontryagin manifold $f^{-1}(\varphi(0)) = \pi^{-1}(0) = N$ is the framed manifold N. □

The two things that are missing for a bijective map $[M, S^p] \to \Omega^{fr}_{p,M}$ is the independence of the Pontryagin construction of the homotopy class of a map $f : M \to S^p$ and the injectivity of the construction. These two properties are stated in the following theorem:

Theorem 2.16. *Two mappings $f : M \to S^p$, $g : M \to S^p$ are smoothly homotopic if and only if the associated Pontryagin manifolds are framed bordant.*

To be able to prove this, we first need the following lemma:

Lemma 2.17. *Let $f, g : M \to S^p$ be smooth maps with a common regular value y. Assume that the framed manifolds $(f^{-1}(y), f^*v)$ and $(g^{-1}(y), f^*v)$ are framed bordant. Then f and g are smoothly homotopic.*

Proof. Set $N := f^{-1}(y)$. Suppose f coincides with g throughout some neighbourhood V of N. Let $h : S^p \setminus \{y\} \to \mathbb{R}^p$ be the stereographic projection. Because $\mathbb{R}^p$ is convex we can define the smooth homotopy

$$H(x,t) := \begin{cases} f(x), & x \in V, \\ h^{-1}(t \cdot h(f(x)) + (1-t) \cdot h(g(x))), & x \in M \setminus V \end{cases}$$

from f to g. Thus it suffices to deform f so that it coincides with g in some small neighbourhood of N without changing $f^{-1}(y)$, so without mapping any new points onto y.

Let $\phi : N \times \mathbb{R}^p \to V \subset M$ be a product neighbourhood of N, where V is a neighbourhood of N small enough so that $f(V) \subset S^p$ and $g(V) \subset S^p$ do not contain the antipode $\overline{y}$ of y. Using the identifications given by the product neighbourhood ϕ and the stereographic projection $h : S^p \setminus \{\overline{y}\} \to \mathbb{R}^p$ we can define

$$F,\ G : N \times \mathbb{R}^p \xrightarrow{\phi} V \xrightarrow{f_{|V},\ g_{|V}} f(V),\ g(V) \subset S^p \setminus \{\overline{y}\} \xrightarrow{h} \mathbb{R}^p$$

with

$$F^{-1}(0) = (f \circ \phi)^{-1}(y) = N \times \{0\} = (g \circ \phi)^{-1}(y) = G^{-1}(0)$$

and

$$D_{(x,0)}F = D_y h \circ D_x f \circ D_{(x,0)}\phi = \mathrm{pr}_{\mathbb{R}^p} = D_y h \circ D_x g \circ D_{(x,0)}\phi = D_{(x,0)}G$$

for all $x \in N$.

Our next aim is to find a constant c such that

$$\langle F(x,u), u \rangle > 0 \qquad \text{and} \qquad \langle G(x,u) \cdot u \rangle > 0$$

for all $x \in N$ and all $u \in \mathbb{R}^p$ with $0 < ||u|| < c$. Then for all u with $||u|| < c$, $F(x,u)$ and $G(x,u)$ will lie in the same open half-space of $\mathbb{R}^p$. In particular, the homotopy

$$(1-t)F(x,u) + tG(x,u)$$

between F and G will not map any new points to 0 for $||u|| < c$.

By Taylor's Theorem,

$$\begin{aligned} F(x,u) &= F(x,0) + D_{(x,0)}F \cdot ((x,u) - (x,0)) + c_f(x,u) \cdot ||(x,0) - (x,u)||^2 \\ &= 0 + u + c_f(x,u) \cdot ||u||^2. \end{aligned}$$

Set $c_f := \limsup_{(x,u):\ ||u||<1} c_f(x,u)$.

Then $||F(x,u) - u|| \leq c_f \cdot ||u||^2$ for $||u|| < 1$ and thus

$$\begin{aligned} &|(F(x,u) - u) \cdot u| \leq c_f ||u||^3 \\ \Rightarrow\ &|F(x,u) \cdot u - ||u||^2| \leq c_f ||u||^3 \\ \Rightarrow\ &F(x,u) \cdot u \geq ||u||^2 - c_f ||u||^3 > 0 \end{aligned}$$

for $0 \leq ||u|| \leq \min\{c_f^{-1}, 1\}$.
Similarly $G(x,u) \cdot u \geq ||u||^2 - c_g ||u^3|| > 0$ for $0 \leq ||u|| \leq \min\{c_g^{-1}, 1\}$. Set $c := \min\{c_f, c_g\}$.

Let $\lambda : \mathbb{R}^p \to \mathbb{R}$ be a smooth map with

$$\lambda(u) = \begin{cases} 1 & \text{for } ||u|| \leq \frac{c}{2}, \\ 0 & \text{for } ||u|| \geq c. \end{cases}$$

Then $H : N \times \mathbb{R}^p \times I \to \mathbb{R}^p$ given by

$$H_t(x,u) = (1 - \lambda(u)t)F(x,u) + \lambda(u)tG(x,u)$$

is a homotopy between $F = H_0$ and a map $H_1 : N \times R^p \to R^p$ that

- coincides with G for $||u|| < \frac{c}{2}$,
- coincides with F for $||u|| \geq c$,
- has no new zeros.

A corresponding deformation of $f_{|V}$ yields a homotopy $\tilde{H}$ from $f_{|V}$ to a map $\tilde{H}_1 : V \to S^p$ that

- coincides with g in a neighbourhood U of N in M,
- coincides with f outside the neighbourhood of U in M,
- has no new points mapping to y.

Then f is smoothly homotopic to $\tilde{H}_1$ which in turn is smoothly homotopic to g. This proves the lemma. □

Proof of Theorem 2.16. "$\Rightarrow$": By Lemma 2.13, $f^{-1}(y)$ and $g^{-1}(y)$ are framed bordant.

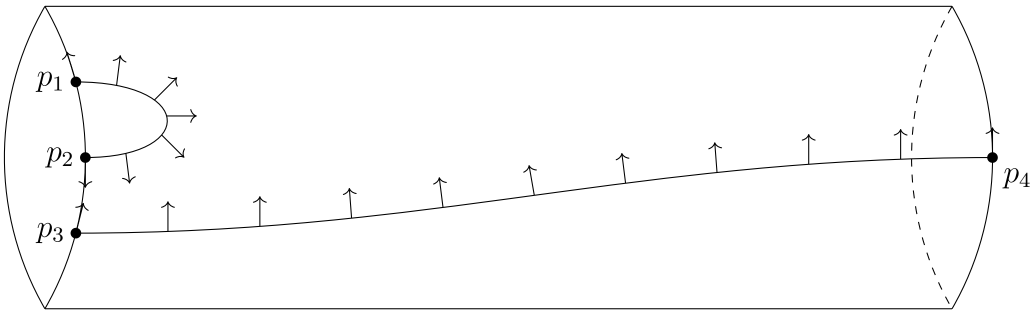
"$\Leftarrow$": If (X, v) is a framed bordism between $f^{-1}(y)$ and $g^{-1}(y)$, then we can construct a homotopy $F : M \times [0,1] \to S^p$ as in Theorem 2.15, whose Pontryagin manifold $(F^{-1}(y), F^*v)$ equals (X, v). Since F_0 and f respectively F_1 and g have the same Pontryagin manifold, $F_0 \sim f$ and $F_1 \sim g$ by Lemma 2.17. Therefore $f \sim g$ by transitivity. □

Example 2.18. We want to use the results from Theorems 2.16 and 2.17 to compute the smooth homotopy classes of smooth maps $S^n \to S^n$ for some $n \in \mathbb{N}$. Theorems 2.6 and 2.7 yield the one-to-one correspondence aimed at in the beginning between framed bordism classes of framed n-dimensional submanifolds of M^m and smooth homotopy classes of smooth maps $M \to S^{m-n}$.

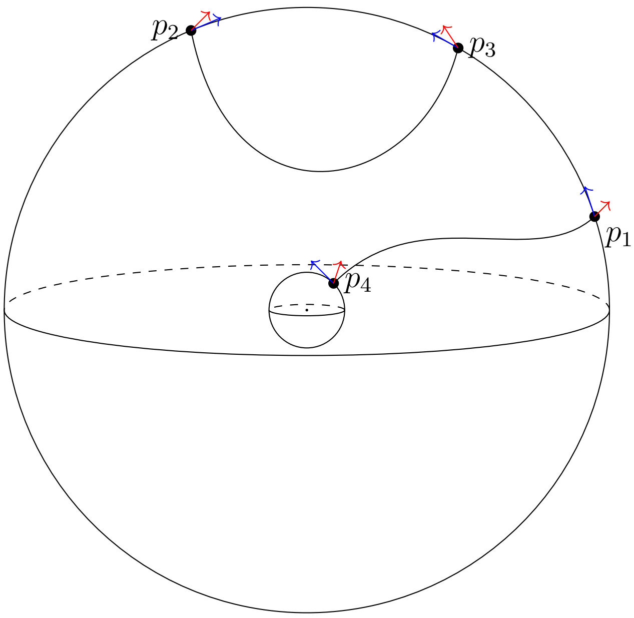
Remark. Using Theorem A.2 this result extends to homotopy classes of continuous maps $M \to S^{m-n}$, not requiring the maps or the homotopies to be smooth.

Let us compute $\Omega^{fr}_{0,S^n} = \{$framed bordism classes of framed 0-dimensional submanifolds of $S^n\}$. A 0-dimensional submanifold of S^n is just a discrete set of points in S^n. A framing of such a submanifold consists of a choice of basis $v(x) \subset (T_xN)^\perp \cong T_xS^n \cong \mathbb{R}^n$.

<u>Case $n = 1$:</u>



<u>Case $n = 2$:</u>



A bordism between two 0-dimensional submanifolds $N_1, N_2 \subset S^n$ is a 1-dimensional submanifold of $S^n \times [0,1]$ with $\partial X = N_1 \times \{0\} \cup N_2 \times \{1\}$ and thus has to be a disjoint union of connected 1-dimensional manifolds connecting two (different) points in N_1, two (different) points in N_2 or connecting a point in N_1 and a point in N_2 as indicated in the pictures above. Not all of these bordisms can be made into framed bordisms though. For this to be possible the framings of the submanifolds at the respective points need to fulfill specific requirements:

- Two points $x_1, x'_1 \in N_1$ or $x_2, x'_2 \in N_2$ in the same submanifold can only be connected by a framed bordism if the orientations of their framings are opposite, i.e. if the change of basis from $v(x_1) \subset (T_{x_1}N_1) \cong \mathbb{R}^n$ to $v(x'_1) \subset (T_{x'_1}N_1) \cong \mathbb{R}^n$ or $w(x_2) \subset (T_{x_2}N_1) \cong \mathbb{R}^n$ to $w(x'_2) \subset (T_{x'_2}N_2) \cong \mathbb{R}^n$ has negative determinant. In the first example above p_1 and p_2 can be connected by a framed bordism while p_1 and p_3 cannot.

- A point $x_1 \in N_1$ and a point $x_2 \in N_2$ can be connected by a framed bordism if the framing $v(x_1)$ of N_1 at x_1 and the framing $w(x_2)$ of N_2 carry the same orientation, i.e. the change of basis has positive determinant. In the first example above p_3 and p_4 can be connected by a framed bordism while p_2 and p_4 cannot.

So suppose N_1 consists of $a_1 + b_1$ points, such that the framing of N_1 is positively oriented in a_1 of those points and negatively oriented in the remaining b_1. Analogously, N_2 is the disjoint union of $a_2 + b_2$ points, in a_1 of which its framing is positively oriented, b_2 points admitting a negatively oriented basis of their respective normal space. Connecting all possible points within N_1 and N_2 by a framed bordism leaves $a_1 - b_1 \in \mathbb{Z}$ "positively oriented points" in N_1 and $a_2 - b_2$ in N_2. Then N_1 and N_2 are framed bordant if and only if $a_1 - b_1 = a_2 - b_2$. Thus, Ω^{fr}_{0,S^n} can be identified with the set of pairs $(a, b) \in \mathbb{N}_0 \times \mathbb{N}_0$ modulo the equivalence relation $(a_1, b_1) \sim (a_2, b_2) \Leftrightarrow a_1 - b_1 = a_2 - b_2$. Recalling the Grothendieck contruction of the integers from the natural numbers one sees that this is the set of integers $\mathbb{Z}$. We can turn the set Ω^{fr}_{0,S^n} into a group by setting the operation to be disjoint union of framed submanifolds. This yields the additive operation we know from $\mathbb{Z}$ given by $(a_1, b_1) + (a_2, b_2) := (a_1 + a_2, b_1 + b_2)$ on the representatives of an equivalence class.

3. Stably Framed Bordism

So far we have only considered submanifolds N of a specific manifold M. We want to remove this restriction and consider arbitrary compact smooth manifolds N. Since we still want to work with the normal bundle of N, we need to embed N into some manifold and define a structure that is independent of this embedding. This is done by first interpreting framings in terms of the normal bundle of N embedded into some sphere S^k for which an embedding is possible and then by "stabilising" this normal bundle. This stabilisation process involves suspensions and considering $S^k \subset S^{k+1}$ embedded into the equator.

The main sources of this chapter are the chapters 6 and 8 of [3]. Most definitions, theorems and lemmata can be found similarly in [3]. Only additional sources are cited explicitly.

We no longer require all our manifolds to be embedded into some $\mathbb{R}^k$. In Lemma 3.3 and Lemma 3.5 we show that for manifolds embedded into some $\mathbb{R}^k$, the definition of framings given in Definition 2.5 is equivalent to the definition given here in terms of the normal bundle.

3.1. Interpretation of Framings in Terms of the Normal Bundle

Definition 3.1. The *normal bundle* $\nu(N \hookrightarrow M)$ of a submanifold $j\colon N \hookrightarrow M$ is the quotient bundle $\nu(N \hookrightarrow M) := (j^*(TM))/(TN)$. If M carries a Riemannian metric, $\nu(N \hookrightarrow M)$ can be considered to be the subbundle $TM_{|N}$ of the tangent bundle of M consisting of the normal spaces $(T_xN)^\perp \subset T_xM$ for each $x \in N$, as described in Section 2.2.

Definition 3.2.
- A *trivialisation* of a vector bundle $p\colon E \to B$ with fibre $\mathbb{R}^n$ is a collection of sections $\{s_i\colon B \to E\}_{1,\dots,n}$ forming a basis of the fibre $E_b = p^{-1}(b)$ pointwise for each $b \in B$.

  Equivalently a trivialisation is a specific bundle isomorphism

$$\begin{array}{ccc} E & \longrightarrow & B \times \mathbb{R}^n \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \mathrm{pr}_B} \\ B & \xrightarrow{id} & B \end{array} .$$

- A *framing* of a vector bundle is a homotopy class of trivialisations, where two trivialisation η and ξ are called *homotopic* if there is a continuous map $F\colon E \times [0,1] \to B \times \mathbb{R}^n$ such that F_t is a trivialisation for every $t \in [0,1]$, $F_0 = \eta$ and $F_1 = \xi$.

- A *normal framing* of a submanifold $N \subset M$ is a homotopy class of trivialisations of the normal bundle $\nu(N \hookrightarrow M)$.

Lemma 3.3. *Let M^m be embedded into $\mathbb{R}^k$ for some $k \in \mathbb{N}$. A framed submanifold $(N^n, v = (v_1, \dots, v_{m-n}))$ of M defines a normal framing of N in M.*

Proof. For each $x \in N$, $\{v_1(x), \dots, v_{m-n}(x)\}$ is a basis of $(T_xN)^\perp \subset T_xM$. For each $x \in N$, define an isomorphism

$$(T_xN)^\perp \xrightarrow{\cong} \mathbb{R}^{m-n}$$

by identifying $v_i(x)$ with the i^{th} standard basis vector e_i of $\mathbb{R}^{m-n}$. By continuity of the framing, this defines a continuous map

$$\nu(N \hookrightarrow M) \to N \times \mathbb{R}^{m-n},$$

an isomorphism in each fibre making the diagram

$$\begin{tikzcd}
\nu(N \hookrightarrow M) \arrow[r] \arrow[d] & N \times \mathbb{R}^{m-n} \arrow[d] \\
N \arrow[r, "id"] & N
\end{tikzcd}$$

commute. □

Definition 3.4. Let N_1^n and N_2^n be two normally framed submanifolds of M^m. Then N_1^n is *normally framed bordant* to N_2^n within M^m if there is a normally framed submanifold $X^{n+1} \subset M^m \times [0,1]$ extending $(N_1^n \times [0,\epsilon)) \cup (N_2^n \times (1-\epsilon, 1])$ with $\partial X^{n+1} = (N_1^n \times \{0\}) \cup (N_2^n \times \{1\})$.

Lemma 3.5. *Let M^m be embedded into $\mathbb{R}^k$ for some $k \in N$. The set of bordism classes of n-dimensional framed submanifolds of M is in bijection with the set of bordism classes of normally framed n-dimensional submanifolds of M.*

Proof. Set $p := m-n$. Let $(N^n, v = (v_1, \dots, v_p))$ be a framed submanifold of M^m. Lemma 3.3 shows how to define a normal framing of N^n in M^m; denote it by ϕ_v.

Now let (N^n, ϕ) be a normally framed submanifold of M^m, that is $\phi\colon \nu(N^n \hookrightarrow M^m) \xrightarrow{\cong} N^n \times \mathbb{R}^p$ is a bundle isomorphism. Using this we define the following framing of N^n in M^m:

$$v_i(x) := (\phi^{-1}(x, e_i)) \in (T_x N^n)^\perp \quad \text{for } x \in N^n.$$

Since ϕ is continuous, an isomorphism in each fibre and $\{e_1, \dots, e_p\}$ forms a basis of $\mathbb{R}^p$, this defines a framing of N in M.

The two constructions are clearly inverse to one another. □

In the following we will use the notation $\Omega^{fr}_{n,M}$ for both the bordism classes of framed submanifolds of M and the bordism classes of normally framed submanifolds of M. Now that we are able to describe framings in terms of the normal bundle, we can look at how to stabilise this normally framed bundle.

3.2. Suspension and the Freundenthal Theorem

Definition 3.6. Define $\mathcal{K}$ to be the category of compactly generated spaces with

- objects: Hausdorff spaces X for which a subset $A \subset X$ is closed if and only if $A \cap C$ is closed for every compact $C \subset X$;
- morphisms: continuous functions between compactly generated spaces.

Let $\mathcal{K}_*$ be the category of compactly generated spaces with *non-degenerate basepoint*, i.e., (X, x_0) is an object of $\mathcal{K}_*$ if $x_0 \hookrightarrow X$ is a neighbourhood deformation retract [3, Ch. 6]. Morphisms in $\mathcal{K}_*$ are basepoint preserving morphisms in $\mathcal{K}$.

By declaring any subset $A \subset X$ to be closed if and only if $A \cap C$ is closed in X for all compact $C \subset X$, any Hausdorff space X can be turned into a compactly generated space $k(X)$. This defines a functor

$$k \colon \mathcal{T}_2 \to \mathcal{K}$$

from the category of Hausdorff spaces $\mathcal{T}_2$ to the category $\mathcal{K}$ of compactly generated spaces.

Remark. The product of two compactly generated spaces X and Y in the category $\mathcal{K}$ is given by $k(X \times Y)$.

Instead of giving $C(X, Y)$ the *compact-open topology* generated by

$$U(K, W) = \{f \in C(X, Y) |\ f(K) \subset W\},$$

where K is compact in X and W is open in Y, we topologise the function space as

$$\mathrm{Map}(X, Y) := K(C(X, Y)),$$

which is also a compactly generated space.

In the following, we will assume all topological spaces to be compactly generated. For details on why we need this restriction see [3, Ch. 6].

Definition 3.7. A space X is called *n-connected* if

$$\pi_k(X) = 0 \text{ for } k \leq n.$$

Definition 3.8. [Some operations on based spaces]

Let (X, x_0) and (Y, y_0) be based topological spaces.

The *wedge product* of X and Y is

$$X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subset X \times Y,$$

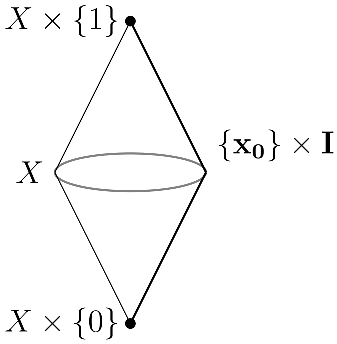
and the *smash product* is the quotient space

$$X \wedge Y = (X \times Y)/(X \vee Y) = (X \times Y)/((X \times \{y_0\}) \cup (\{x_0\} \times Y)).$$

The *reduced suspension* of (X, x_0) is the quotient space

$$\Sigma X = (X \times I)/((X \times \{0, 1\}) \cup \{x_0\} \times I) = S^1 \wedge X$$

(see the following picture).



Lemma 3.9. *The reduced suspension Σ is functorial with respect to based maps $f\colon (X, x_0) \to (Y, y_0)$.*

Proof. Let (X, x_0), (Y, y_0) and (Z, z_0) be based spaces, $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ based maps. $\Sigma\colon Top_* \to Top_*$ is defined as

$$\Sigma X = (X \times I)/(\{(X \times \{0, 1\}) \cup (x_0 \times I))$$

on based spaces and as the quotient map of

$$f \times \mathrm{id}\colon X \times I \to Y \times I$$

on based maps. It is well-defined because $X \times \{0, 1\}$ is mapped to $Y \times \{0, 1\}$ and $\{x_0\} \times I$ is mapped to $\{y_0\} \times I$ since f is based.

Functoriality of Σ:

- $\Sigma(\mathrm{id}_X)\colon \Sigma X \to \Sigma X$ is the quotient map of $\mathrm{id}_X \times\ \mathrm{id}_I\colon X \times I \to X \times I$ by the same quotient on the domain and codomain and thus equals $\mathrm{id}_{\Sigma X}$.
- $\Sigma(g \circ f)\colon \Sigma X \to \Sigma Z$ is the quotient map of $(g \circ f) \times \mathrm{id}_I\colon X \times I \to Z \times I$ by $(X \times \{0, 1\}) \cup (x_0 \times I)$ and $(Z \times \{0, 1\}) \cup (z_0 \times I)$. $\Sigma(g) \circ \Sigma(f)$ is the concatenation of the quotient maps of $f \times \mathrm{id}_I$ and $g \times \mathrm{id}_I$. Since the collapsed part $(Y \times \{0, 1\}) \cup (y_0 \times I)$ in the image of $\Sigma(f)$ gets mapped to $(Z \times \{0, 1\}) \cup (z_0 \times I)$ by $\Sigma(g)$, $\Sigma(g \circ f) = \Sigma(g) \circ \Sigma(f)$. □

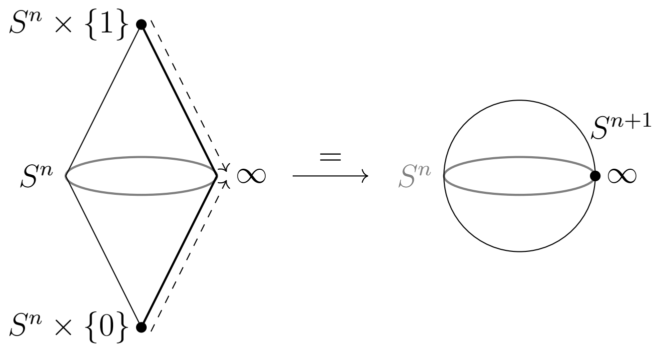
Remark. In particular, the suspension defines a map

$$\Sigma\colon [X, Y]_0 \to [\Sigma X, \Sigma Y]_0,$$

where $[X, Y]_0$ denotes the based homotopy classes of based maps from X to Y.

Proposition 3.10. *The reduced suspension ΣS^n of the n-sphere is homeomorphic to S^{n+1}.*

Proof. The following picture indicates a homeomorphism:



□

Notation. Denote the homeomorphism by $s_n\colon \Sigma S^n \to S^{n+1}$.

Corollary 3.11. *The k-fold suspension $\Sigma^k(S^n)$ is homeomorphic to S^{k+n}.* □

By the above remark, the suspension defines a map

$$\Sigma\colon [S^k, Y]_0 \to [S^{k+1}, \Sigma Y]_0,$$

which turns out to be a homomorphism

$$\Sigma\colon \pi_k(Y) \to \pi_{k+1}(\Sigma Y).$$

for any based space Y. For $Y = S^n$ we obtain

$$\Sigma\colon \pi_k(S^n) \to \pi_{k+1}(S^{n+1}).$$

One naturally considers $Y \subset \Sigma Y$ embedded as $Y \times \{\frac{1}{2}\}$. For the sphere this yields the identification of $S^k \subset \Sigma S^k = S^{k+1}$ with the equator as indicated in the picture above. Using this we can interpret the suspension map Σ in terms of framed bordism:

If $f\colon S^k \to S^n$ is smooth, then $\Sigma(f)\colon \Sigma S^k = S^{k+1} \to S^{n+1} = \Sigma S^n$ is smooth away from the basepoints, being the same map on the equator $f = \Sigma(f)_{|S^k}\colon S^k \to S^n$. If $y \in S^n$ is not the basepoint and a regular value for f, it is also a regular value for $\Sigma(f)$. Hence $N := f^{-1}(y)$ is a submanifold of S^k, $y \in S^{n+1}$ lies in the equator of S^{n+1} and $\Sigma(f)$ maps the equator S^k of S^{k+1} to the equator S^n of S^{n+1}. Thus $(\Sigma(f))^{-1}(y) = N \subset S^k \subset S^{k+1}$.

By the Pontryagin-Thom construction N is a framed submanifold of S^k and S^{k+1}. Let us look at how these two framings compare:

$$\begin{aligned}
\nu(N \hookrightarrow S^{k+1}) &= \nu(N \hookrightarrow S^k) \oplus \nu(S^k \hookrightarrow S^{k+1})_{|N} \\
&= \nu(N \hookrightarrow S^k) \oplus \epsilon_N \text{ and} \\
\nu(y \hookrightarrow S^{n+1}) &= \nu(y \hookrightarrow S^n) \oplus \epsilon_{\{y\}},
\end{aligned}$$

where $\epsilon_N = N \times \mathbb{R}$ and $\epsilon_{\{y\}} = \{y\} \times \mathbb{R}$ are the trivial 1-dimensional bundles over N and $\{y\}$.

As described above $\Sigma(f)_{|S^k}$ corresponds to f. Locally near the equator $S^k \subset S^{k+1}$, $\Sigma(f)$ has the form

$$f \times \mathrm{id}\colon S^k \times (-\epsilon, \epsilon) \to S^n \times (-\epsilon, \epsilon),$$

so the differential $D_x\Sigma(f)\colon T_xS^{k+1} \to T_yS^{n+1}$, $x \in V$, maps $\epsilon_{\{x\}}$ identically onto $\epsilon_{\{y\}}$.

This shows the following:

Theorem 3.12. *Taking the Pontryagin manifold $(\Sigma(f))^{-1}(y)$ of a suspended map $f\colon S^k \to S^n$ yields the same manifold $(\Sigma(f))^{-1}(y) = f^{-1}(y)$ embedded into the equator $S^k \subset S^{k+1}$, with framing corresponding to the direct sum of the old framing and the trivial 1-dimensional framing.* □

Iterating this process we see:

Corollary 3.13. *Taking the Pontryagin manifold of an l-fold suspended map*

$$\Sigma^l(f)\colon S^{k+l} \to S^{n+l}$$

yields the same manifold $((\Sigma^l(f)))^{-1}(y) = f^{-1}(y)$ embedded into the iterated equator $S^k \subset S^{k+l}$ with new framing

$$\nu_{new} = \nu_{old} \oplus \epsilon^l_V.$$

Theorem 3.14. *[Freundenthal suspension theorem]*
Let X be an $(n-1)$-connected space with $n \geq 2$. Then the suspension homomorphism

$$\Sigma\colon \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k < 2n-1$ and an epimorphism for $k = 2n-1$.

Proof. See [3, Ch. 10]. □

This allows us to make the following definition:

Definition 3.15. The k^{th} *stable homotopy group* of a based space X is the colimit

$$\pi_k^S(X) := \operatorname{colim}_{l\to\infty} \pi_{k+l}(\Sigma^l X)$$

over the homomorphisms $\Sigma\colon \pi_k(\Sigma^l X) \to \pi_{k+1}(\Sigma^{l+1} X)$.
The *stable k-stem* is

$$\pi_k^S := \pi_k^S(S^0).$$

Corollary 3.16. *If X is path-connected, then*

$$\pi_k^S(X) = \pi_{2k}(\Sigma^k X) = \pi_{k+l}(\Sigma^l X) \quad \textit{for } l \geq k.$$

For the stable k-stem,

$$\pi_k^S = \pi_{2k+2}(S^{k+2}) = \pi_{k+l}(S^l) \quad \textit{for } l \geq k+2.$$

Corollary 3.17. *The Pontryagin-Thom construction defines an isomorphism*

$$\pi_k^S \xrightarrow{\cong} \Omega^{fr}_{k,S^n}$$

for any $n \geq 2k+2$.

Proof. Let $n \geq 2k+2$ and set $l := n-k$. Then by Section 2 and the previous corollary:

$$\pi_k^S \cong \pi_{k+l}(S^l) = [S^{k+l}, S^l]_0 = [S^n, S^{n-k}]_0 \cong \Omega^{fr}_{k,S^n}. \qquad \square$$

Finally we want to remove the restriction of our manifolds being submanifolds of some sphere. This can be done by defining the so called *stable normal framing*:

Definition 3.18. A *stable normal framing* of a manifold N^n, is an equivalence class of trivialisations $\nu(N^n \hookrightarrow S^k) \oplus \epsilon^l \xrightarrow{\cong} N \times \mathbb{R}^{k-n+l}$ corresponding to some embedding $i_k\colon N \hookrightarrow S^k$ into a sphere, subject to the following equivalence relation:
$(i_{k_1}\colon N \hookrightarrow S^{k_1}, \nu \oplus \epsilon^{l_1} \cong N \times \mathbb{R}^{k_1-n+l_1}) \sim (i_{k_2}\colon N \hookrightarrow S^{k_2}, \nu \oplus \epsilon^{l_2} \cong N \times \mathbb{R}^{k_2-n+l_2})$ if there is some K greater that k_1 and k_2 such that the direct sum trivialisations

$$\nu \oplus \epsilon^{l_1} \oplus \epsilon^{K-k_1-l_1} \cong \epsilon^{k_1-n+l_1} \oplus \epsilon^{K-k_1-l_1} = \epsilon^{K-n}$$

and

$$\nu \oplus \epsilon^{l_2} \oplus \epsilon^{K-k_2-l_2} \cong \epsilon^{k_2-n+l_2} \oplus \epsilon^{K-k_2-l_2} = \epsilon^{K-n}$$

are homotopic.

Corollary 3.19. *The stable k-stem π_k^S is isomorphic to the stably normally framed bordism classes of stably normally framed k-dimensional smooth, oriented compact manifolds without boundary.*

We have now given a description of the stable k-stem π_k^S in terms of stably framed bordism. The next step is to express $\pi_k^S(X)$ in terms of bordism. The structure we need to add is to consider only manifolds *in* X, meaning manifolds N with a specified mapping $g\colon N \to X$ to X. Then we can define stably framed bordism in X:

Definition 3.20. Let $(N_i^n, \gamma_i)_{i\in\{1,2\}}$ be two stably framed manifolds and $g_i\colon N_i \to X$, $i \in \{1,2\}$ continuous maps. Then (N_0, γ_0, g_0) is called *stably framed bordant* to (N_1, γ_1, g_1) *in* X if there is a stably framed bordism (W, Γ) between (N_0, γ_0) and (N_1, γ_1) and a map $G\colon W \to X$ restricting to g_0 and g_1 on N_0 repectively N_1. We say that (W, Γ, G) restricts to $(N_0 \coprod N_1, \gamma_0 \coprod \gamma_1, g_0 \coprod g_1)$.

Notation. Set $X_+ := X \coprod \{\infty\}$ to be the disjoint union of X and a point, ∞ being the new basepoint of X_+. Let $\Omega_n^{fr}(X)$ denote the stably framed bordism classes of n-dimensional stably framed manifolds in X.

Since every manifold maps uniquely to a point and $S^0 = pt_+$, the previous corollary can be restated as:

$$\Omega_n^{fr}(pt) = \pi_n^S(pt_+).$$

The natural next step is to prove

Theorem 3.21.

$$\Omega_n^{fr}(X) = \pi_n^S(X_+).$$

However, we will not prove this here, but prove an even more general version in the next chapter.

4. General Bordism Theories

In this section we describe arbitrary bordism theories that lead to generalised homology theories. These bordism theories are more general than the stably framed bordisms in a manifold X we studied above because we allow more general structures than just framings. We will still consider manifolds in some specified manifold X but this time carrying some different, more general, stable structure on the stable normal bundle. This structure will be given by a sequence of groups and homormorphisms, called **G**-structure. The generalised homology theories arise from so called *spectra*, which we will define next.

The main sources of this chapter are chapter 8 of [3] and chapter 3 of [1]. Only additional sources are given explicitely.

4.1. Spectra

Definition 4.1. A *spectrum* is a sequence of pairs $\mathbf{K} = \{K_n, k_n\}$ of pointed topological spaces K_n and pointed continuous maps

$$k_n \colon \Sigma K_n = S^1 \wedge K_n \to K_{n+1},$$

where ΣK_n denotes the reduced suspension of K_n.

Example 4.2. [Suspension spectrum]
Let X be a pointed topological space. Set

$$K_n := \{pt\} \text{ for } n < 0 \text{ and } K_n := \Sigma^n X = S^n \wedge X \text{ for } n \geq 0$$

with

$$k_n \colon \Sigma K_n = \Sigma(S^n \wedge X) = \Sigma S^n \wedge X \xrightarrow{s_n \wedge id_X} S^{n+1} \wedge X = K_{n+1}.$$

Then (K_n, k_n) is a spectrum.
For $X = pt_+$, $\Sigma^n pt_+ = S^n$, so $(K_n, k_n) = (S^n, s_n \colon \Sigma S^n \to S^{n+1})$. This spectrum is called the *sphere spectrum*.

Using the identification $\Sigma^n X = S^n \wedge X$, the definition of stable homotopy groups can be rewritten as

$$\pi_n^S(X) = \operatorname{colim}_{l \to \infty} \pi_{n+l}(S^l \wedge X),$$

where the colimit is taken over the composite homomorphisms

$$\pi_{n+l}(S^l \wedge X) \xrightarrow{\Sigma} \pi_{n+l+1}(\Sigma(S^l \wedge X)) \xrightarrow{k_l} S^{l+1} \wedge X.$$

Example 4.3. [Eilenberg-MacLane spectrum]
Let π be an abelian group. An *Eilenberg-MacLane space of type* $K(\pi, n)$ is a CW-complex $K(\pi, n)$ such that $\pi_n(K(\pi, n)) = \pi$ and $\pi_k(K(\pi, n)) = 0$ for all $k \neq n$. In the case $n = 1$, π may be non-abelian and in the case $n = 0$, we think of $K(\pi, n)$ with the discrete topology. Eilenberg-MacLane spaces exist.
The Eilenberg-MacLane spectrum for an abelian group π consists of the spaces $K(\pi, n)$ and inclusions of subcomplexes $k_n \colon \Sigma K(\pi, n) \to K(\pi, n+1)$ obtained by attaching cells to $\Sigma K(\pi, n)$. This gives us the Eilenberg-MacLane spectrum

$$\mathbf{K}(\pi) = \{K(\pi, n), k_n\}.$$

Taking $\pi = \mathbb{Z}$, the Eilenberg-MacLane spectrum can be used to define ordinary homology and cohomology [3]:

$$H_n(X;\mathbb{Z}) = \operatorname{colim}_{l\to\infty} \pi_{n+l}(X_+ \wedge K(\mathbb{Z},l)),$$
$$H^n(X;\mathbb{Z}) = \operatorname{colim}_{l\to\infty} [\Sigma^l X_+, K(\mathbb{Z}, n+l)]_0.$$

This motivates the following definition of homology with respect to any spectrum, which can be done analogously for cohomology.

Definition 4.4. Let $\mathbf{K} = \{K_n, k_n\}$ be a spectrum. Define the

- n^{th} *(unreduced) homology with coefficients in* $\mathbf{K}$ to be the functor

  $$H_n\colon Top \to Ab,\ H_n(X;\mathbf{K}) = \operatorname{colim}_{l\to\infty} \pi_{n+l}(X_+ \wedge K_l),$$

  where the colimit is taken over the homomorphisms

  $$m_l\colon \pi_{n+l}(X_+ \wedge K_l) \xrightarrow{\Sigma} \pi_{n+l+1}(\Sigma(X_+ \wedge K_l)) \to \pi_{n+l+1}(X_+ \wedge \Sigma K_l) \xrightarrow{\mathrm{id}\wedge k_l} X_+ \wedge K_{l+1}.$$

- n^{th} *reduced homology with coefficients in* $\mathbf{K}$ to be the functor

  $$\tilde{H}_n\colon Top_* \to Ab,\ \tilde{H}_n(X;\mathbf{K}) = \operatorname{colim}_{l\to\infty} \pi_{n+l}(X \wedge K_l),$$

  where the colimit is taken over the homomorphisms

  $$m_l\colon \pi_{n+l}(X \wedge K_l) \xrightarrow{\Sigma} \pi_{n+l+1}(\Sigma(X \wedge K_l)) \to \pi_{n+l+1}(X \wedge \Sigma K_l) \xrightarrow{\mathrm{id}\wedge k_l} X \wedge K_{l+1}.$$

4.2. G-structure

Let $M^k \hookrightarrow S^{k+n}$ be a submanifold of codimension n and $G \to O(n)$ a continuous group homomorphism from a topological group G to the orthogonal group $O(n)$. A topological group is a Hausdorff space G together with a group structure such that both $*\colon G\times G \to G$ and ${}^{-1}\colon G \to G$ are continuous.

For each topological group G let

$$\begin{array}{c} EG \\ \downarrow{\scriptstyle P} \\ BG \end{array}$$

denote the *universal principal G-bundle.* Then, up to isomorphism, any principal G-bundle over a paracompact space B arises as the pullback of $EG \to BG$. The space BG is called *classifying space* for G and the map c along which the pullback is taken:

$$\begin{array}{ccc} E & \longrightarrow & EG \\ \downarrow & & \downarrow{\scriptstyle P} \\ B & \xrightarrow{c} & BG \end{array}$$

is called *classifying map* for the principal G-bundle $E \to B$. For details on classifying spaces and universal bundles see the appendix or [1,3].

Note. Using the Gram-Schmidt orthonogalisation process it can be shown that the orthogonal group $O(n)$ is a strong deformation retract of the general linear group $GL_n(\mathbb{R})$. This implies that the isomorphism classes of n-dimensional vector bundles stand in bijection with the isomorphism classes of $\mathbb{R}^n$-bundles with structure group $O(n)$. Since an $\mathbb{R}^n$-bundle with structure group $O(n)$ carries a metric, we can henceforth assume all our $\mathbb{R}^n$-bundles to have structure group $O(n)$ and carry a metric.

Definition 4.5. A *(normal) G-structure* on $M^k \hookrightarrow S^{k+l}$ is a pullback square

$$\begin{tikzcd}
\nu(M^k \hookrightarrow S^{k+l}) \arrow[r, "\gamma"] \arrow[d] & EG \times_G \mathbb{R}^l \arrow[d] \\
M^k \arrow[r, "c"] & BG
\end{tikzcd},$$

where $EG \times_G \mathbb{R}^l \to BG$ is the $\mathbb{R}^l$-fibre bundle associated with the principal G-bundle.

Now let $\mathbf{G} = \{G_l, g_l, i_l\}$ be a sequence of topological groups with continuous homomorphisms

$$i_l \colon G_l \to G_{l+1}, \qquad g_l \colon G_l \to O(l)$$

such that for each l the following diagram commutes:

$$\begin{tikzcd}
G_l \arrow[r, "i_l"] \arrow[d, "g_l"] & G_{l+1} \arrow[d, "g_{l+1}"] \\
O(l) \arrow[r, hook, "\text{incl.}"] & O(l+1)
\end{tikzcd}$$

where incl.: $O(l) \to O(l+1)$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 4.6. A *(normal) stable* **G***-structure* on M^k consists of a G_l-structure on $M^k \hookrightarrow S^{k+l}$, a G_{l+1}-structure on $M^k \hookrightarrow S^{k+l+1}$, and so forth, such that the following diagram commutes for every $l \geq l_0$ for some $l_0 \in \mathbb{N}$:

$$\begin{tikzcd}
\nu(M^k \hookrightarrow S^{k+l}) \arrow[rr, "\gamma_l"] \arrow[dr] \arrow[dd, "\mathrm{id} \oplus \epsilon_M"'] & & EG_l \times_{G_l} \mathbb{R}^l \arrow[dr] \arrow[dd] & \\
& M^k \arrow[rr, "c_l"'] \arrow[dd, "="] & & BG_l \arrow[dd] \\
\nu(M^k \hookrightarrow S^{k+l+1}) \arrow[rr, "\gamma_{l+1}"] \arrow[dr] & & EG_{l+1} \times_{G_{l+1}} \mathbb{R}^{l+1} \arrow[dr] & \\
& M^k \arrow[rr, "c_{l+1}"] & & BG_{l+1}
\end{tikzcd}$$

The vertical maps $EG_l \times_{G_l} \mathbb{R}^l \to EG_{l+1} \times_{G_{l+1}} \mathbb{R}^l$ and $BG_l \to BG_{l+1}$ are induced by i_l and g_l.

Definition 4.7. Given a **G**-structure $\mathbf{G} = \{G_l, g_l, i_l\}$, define the k^{th} **G**-bordism group $\Omega_k^G(X)$ of a manifold X to be the **G**-bordism classes of closed (compact and boundaryless) k-dimensional manifolds (M, f) in X with stable **G**-structure γ on the normal bundle of an embedding $j \colon M^k \hookrightarrow S^{k+l}$ in a sphere.

An element $[M^k, f, \gamma] \in \Omega_k^G(X)$ is represented by a triple (M^k, f, γ) with

- M^k a k-dimensional closed manifold,
- $f\colon M^k \to X$ a continuous map,
- $\gamma\colon \nu(M^k \hookrightarrow S^{k+l}) \to EG_l \times_{G_l} \mathbb{R}^l$ a G_l-structure on the normal bundle $\nu(M^k \hookrightarrow S^{k+l})$.

G-bordism is the equivalence relation generated by

- $(M^k \hookrightarrow S^{k+l}, f, \gamma) \sim (M^k \hookrightarrow S^{k+l+1}, f, \gamma')$ if γ and γ' fit into a commutative diagram as in Definition 4.6;
- $(M_0^k \hookrightarrow S^{k+l}, f_0, \gamma_0) \sim (M_1^k \hookrightarrow S^{k+l}, f_1, \gamma_1)$ if there is a compact submanifold $W^{k+1} \subset S^{k+l} \times I$, a map $F\colon W^{k+1} \to X$ and a stable **G**-structure Γ on $\nu(W^{k+1} \hookrightarrow S^{k+l} \times I)$ such that

$$(\partial W^{k+1}, F_{|\partial W^{k+1}}, \Gamma_{|\partial W^{k+1}}) = (M_0^k \coprod M_1^k, f_0 \coprod f_1, \gamma_0 \coprod \gamma_1).$$

Examples of **G**-structures:

Example 4.8. [Empty structure]
The most basic example of a stable **G**-structure on a manifold M^k is requiring no structure in addition to the orthogonal structure we already have on normal bundles. This means that all groups G_l are simply the orthogonal groups $O(l)$ with maps $g_l = \mathrm{id}_{O(l)}$ and $i_l\colon O(l) \hookrightarrow O(l+1)$ the inclusions of a matrix $A \in O(l)$ into the top left corner as

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in O(l+1).$$

Example 4.9. [Framing]
For a stably framed normal bundle, we require our normal bundles to be stably trivial. This corresponds to the reduction of the structure groups of the normal bundles to the trivial group. In terms of a **G**-structure, this is expressed by

$$G_l = \mathbf{1},\ g_l\colon \mathbf{1} \hookrightarrow O(l),\ i_l = \mathrm{id}\,.$$

Example 4.10. [Orientation]
An orientation is weaker than a framing but stronger than the empty structure. In addition to the orthogonal structure we require that the orientation of the normal bundle be preserved under transition functions. This corresponds to the reduction of the structure group to the special orthogonal group:

$$G_l = SO(l),\ g_l\colon SO(l) \hookrightarrow O(l),\ i_l\colon SO(l) \hookrightarrow SO(l+1),\ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

4.3. Thom Space

Definition 4.11. Given a vector bundle $p\colon E \to B$, define the *disc* and *sphere bundle* of p to be

- $D(p)\colon D(E) \to B, x \mapsto p(x)$ with $D(E) = \{x \in E \mid ||x|| \leq 1\} \subset E$, respectively
- $S(p)\colon S(E) \to B, x \mapsto p(x)$ with $S(E) = \{x \in E \mid ||x|| = 1\} \subset D(E) \subset E$.

Define the *Thom space* of p to be the quotient space

$$\mathrm{Th}(E) := D(E)/S(E).$$

Remark. For a compact base space, the Thom space is homeomorphic to the one-point compactification of the total space:

$$\phi\colon D(E) \setminus S(E) \to E, x \mapsto \frac{x}{\sqrt{1-||x||^2}}$$

is a diffeomorphism fiberwise and a homeomorphism globally. The one-point compactification of $D(E) \setminus S(E)$ is homeomorphic to $D(E)/S(E)$. So

$$D(E)/S(E) \cong (D(E) \setminus S(E))^c \cong E^c,$$

where X^c denotes the one-point compactification of a locally compact topological space X.

Note that in the case of a compact base space, the total space is locally compact and so its one-point compactification is compact.

Remark. The 0-section $s\colon B \to E,\ b \mapsto 0_b \in p^{-1}(b) \subset D(E) \setminus S(E) \subset E$ defines an embedding of B into the Thom space $\mathrm{Th}(E) = D(E)/S(E)$.

Let

$$\epsilon(B) := \epsilon\colon B \times \mathbb{R} \xrightarrow{\mathrm{pr}_B} B$$

denote the 1-dim trivial vector bundle B. More generally, let

$$\epsilon^k(B) := \epsilon^k\colon B \times \mathbb{R}^k \xrightarrow{\mathrm{pr}_B} B$$

denote the k-dim trivial vector bundle over B.

Lemma 4.12. *For a vector bundle $p\colon E \to B$, the Thom space $\mathrm{Th}(E \oplus \epsilon^k)$ is homeomorphic to the k-fold reduced suspension $\Sigma^k(\mathrm{Th}(E))$.*

Proof. Let $\varphi\colon D^{n+1} \to D^n \times I$ be an $O(n)$-equivariant homeomorphism. It induces a homeomorphism

$$D(E \oplus \epsilon) \to D(E) \times I,\ (v, x) \mapsto \varphi(v, x),$$

which in turn induces

$$\begin{aligned} \mathrm{Th}(E \oplus \epsilon) &= (D(E \oplus \epsilon))/(S(E \oplus \epsilon)) \to (D(E) \times I)/((S(E) \times I \cup D(E) \times \{0,1\})) \\ &= \Sigma((D(E))/(S(E))) \\ &= \Sigma(\mathrm{Th}(E)). \end{aligned}$$

Iterating this process we see that for any $k \in \mathbb{N}$, $\mathrm{Th}(E \oplus \epsilon^k) = \Sigma^k(\mathrm{Th}(E))$. □

Lemma 4.13. *A vector bundle map*

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & E' \arrow[d] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

which is a metric-preserving isomorphism in each fibre induces a map of Thom spaces

$$\mathrm{Th}(E) \to \mathrm{Th}(E').$$

Proof. Because $\tilde{f}\colon E \to E'$ and $f\colon B \to B'$ are metric preserving, they induce $\tilde{f}_{|D(E)}\colon D(E) \to D(E')$ and $\tilde{f}_{|S(E)}\colon S(E) \to S(E')$. Thus

$$\mathrm{Th}(f)\colon (D(E))/(S(E)) \to (D(E'))/(S(E')),\ [x] \mapsto [\tilde{f}(x)]$$

is well-defined. □

4.4. Thom Spectrum

Now let a **G**-structure $\{G_l, g_l, i_l\}$ be given. Recall that the G_l are topological groups and g_l and i_l are continuous homomorphisms

$$i_l\colon G_l \to G_{l+1}, \qquad\qquad g_l\colon G_l \to O(l)$$

such that for each l the following diagram commutes:

$$\begin{tikzcd} G_l \arrow[r, "i_l"] \arrow[d, "g_l"] & G_{l+1} \arrow[d, "g_{l+1}"] \\ O(l) \arrow[r, hook, "incl."] & O(l+1) \end{tikzcd} .$$

The homomorphism $g_l\colon G_l \to O(l) \subset \mathrm{GL}_l(\mathbb{R})$ induces an action of G_l on $\mathbb{R}^l$, so we can form the associated $\mathbb{R}^l$-bundle with structure group G_l over BG_l:

$$\begin{tikzcd} V_l := EG_l \times_{G_l} \mathbb{R}^l \arrow[d, "Q_l"] \\ BG_l \end{tikzcd} .$$

Thanks to the homomorphism $g_l\colon G_l \to O(l)$ we can also regard $V_l \to BG_l$ as an $\mathbb{R}^l$-bundle with structure group $O(l)$. Thus, $V_l \xrightarrow{Q_l} BG_l$ carries a metric, and the Thom space $MG_l := \mathrm{Th}(V_l) = (D(V_l))/(S(V_l))$ is defined. Functoriality of the universal bundles $EG_l \to BG_l$ induces bundle maps

$$\begin{tikzcd} EG_l \arrow[r, "Ei_l"] \arrow[d, "P_l"] & EG_{n+1} \arrow[d, "P_{l+1}"] \\ BG_l \arrow[r, "Bi_l"] & BG_{l+1} \end{tikzcd}$$

which extend to

$$\begin{tikzcd} EG_l \times_{G_l} \mathbb{R}^l = V_l \arrow[r, "Vi_l"] \arrow[d, "Q_l"] & V_{l+1} = EG_{l+1} \times_{G_{l+1}} \mathbb{R}^{l+1} \arrow[d, "Q_{l+1}"] \\ BG_l \arrow[r, "Bi_l"] & BG_{l+1} \end{tikzcd} ,$$

where $Vi_l\colon V_l \to V_{l+1}$ is a linear injection on each fibre.

Theorem 4.14. *For a* $\mathbf{G}$*-structure* $\mathbf{G} = \{G_l, g_l, i_l\}$ *and the bundles* $Q_l \colon V_l \to BG_l$*, the fiberwise injection* $V_l \to V_{l+1}$ *described above extends to a metric preserving bundle map*

$$\begin{CD} V_l \oplus \epsilon @>>> V_{l+1} \\ @VVV @VVV \\ BG_l @>>> BG_{l+1} \end{CD}$$

which is an isomorphism in each fibre and hence induces a map of Thom spaces

$$\mathrm{Th}(V_l \oplus \epsilon) = \Sigma(\mathrm{Th}(V_l)) = \Sigma(MG_l) \to MG_{l+1} = \mathrm{Th}(V_{l+1})$$

denoted by

$$k_l \colon \Sigma(MG_l) \to MG_{l+1}.$$

Thus, $\mathbf{MG} = \{MG_l, k_l\}$ *is a spectrum, called the* Thom spectrum *of* $\mathbf{G}$*.*

Proof. Consider the following pullback square:

$$\begin{CD} \gamma_l^*(V_{l+1}) @>>> V_{l+1} \\ @VVV @VVV \\ BG_l @>{Bi_l := \gamma_l}>> BG_{l+1} \end{CD}.$$

Recalling the above commutative square induced by i_l on V_l and BG_l and using the universal property of pullbacks, we obtain a bundle map $\phi \colon V_l \to \gamma_l^*(V_{l+1})$ that fits into the following commutative diagram:

$$\begin{tikzcd} V_l \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "Vi_l"] \arrow[ddr, bend right, "Q_l"'] & & \\ & \gamma_l^*(V_{l+1}) \arrow[r, "\overline{\gamma_l}"] \arrow[d, "\overline{Q_{l+1}}"] & V_{l+1} \arrow[d, "Q_{l+1}"] \\ & BG_l \arrow[r, "Bi_l =: \gamma_l"] & BG_{l+1} \end{tikzcd}.$$

Since Vi_l is a linear injection in each fibre and $\overline{\gamma_l}$ is an isomorphism in each fibre, ϕ is also a linear injection in each fibre and thus

$$\begin{CD} V_l @>{\phi}>> \phi(V_l) \\ @V{Q_l}VV @VV{\overline{Q_{l+1}}_{|\phi(V_l}}V \\ BG_l @>{\mathrm{id}}>> BG_l \end{CD}$$

is an isomorphism of vector bundles.

Set ξ to be the orthogonal complement of $\phi(V_l)$ in $\gamma_l^*(V_{l+1})$. Then $\gamma_l^*(V_{l+1}) = \phi(V_l) \oplus \xi$, where $\xi \colon E(\xi) \to BG_l$ is a 1-dimensional $\mathbb{R}^l$-bundle. We now want to understand why ξ is the trivial bundle:

Recall that $V_l = EG_l \times_{G_l} \mathbb{R}^l$, where G_l acts on $\mathbb{R}^l$ via $g_l \colon G_l \to O(l)$; similarly for V_{l+1}. Because

$$\begin{CD} G_l @>{i_l}>> G_{l+1} \\ @V{g_l}VV @VV{g_{l+1}}V \\ O(l) @>>> O(l+1) \end{CD}$$

commutes, the image of G_l under $g_{l+1} \circ i_l$ lies in $O(l) \subset O(l+1)$ naturally seen as a subset, only acting on $\mathbb{R}^l \subset \mathbb{R}^{l+1}$. Looking more closely into the definition of fibre bundles, we see that the transition functions of $\gamma_l^*(V_{l+1})$ are of the form

$$\theta_{\varphi,\varphi'} \colon U \to O(l+1) \text{ for } U \subset BG_l,\ \varphi, \varphi' \colon U \times \mathbb{R}^{l+1} \to \overline{Q_{l+1}}^{-1}(U)$$

with $\mathrm{im}(\theta_{\varphi,\varphi'}) \subset O(l)$ acting invariantly on $\phi(V_l)$, leaving the orthogonal complement ξ fixed. The transition functions are thus trivial on the 1-dimensional bundle ξ; hence $\xi = \epsilon = BG_l \times \mathbb{R}$.

So the above diagram can be written as

$$\begin{tikzcd}
V_l \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "Vi_l"] \arrow[ddr, bend right, "Q_l"'] & & \\
& \phi(V_l) \oplus \epsilon \arrow[r, "\overline{\gamma_l}"] \arrow[d] & V_{l+1} \arrow[d] \\
& BG_l \arrow[r, "Bg_l := \gamma_l"] & BG_{l+1}
\end{tikzcd}$$

inducing the bundle map

$$\begin{tikzcd}
V_l \oplus \epsilon \arrow[r, "\overline{\gamma_l} \circ (\phi \oplus \mathrm{id})"] \arrow[d] & V_{l+1} \arrow[d] \\
BG_l \arrow[r] & BG_{l+1}
\end{tikzcd} .$$

This bundle map extends the fiberwise injection $Vi_l \colon V_l \to V_{l+1}$ because $\overline{\gamma_l} \circ \phi = Vi_l$.

Viewing $V_l \oplus \epsilon$ and V_{l+1} as $\mathbb{R}^{l+1}$-bundles with structure group $O(l+1)$, we see that the isomorphism on each fibre is given by the action of an element in $O(l+1)$ and thus preserves the metric. The previous two lemmata give us the following map:

$$k_l \colon \mathrm{Th}(V_l \oplus \epsilon) = \Sigma(\mathrm{Th}(V_l)) = \Sigma(MG_l) \to MG_{l+1} = \mathrm{Th}(V_{l+1}). \qquad \square$$

4.5. Thom's Theorem

Theorem 4.15. *The bordism groups $\Omega_k^{\mathbf{G}}(X)$ are isomorphic to*

$$H_k(X; \mathbf{MG}) = \mathrm{colim}_{l\to\infty}\, \pi_{k+l}(X_+ \wedge MG_l).$$

Proof. To prove this theorem, we define an isomorphism

$$d \colon \mathrm{colim}_{l\to\infty}\, \pi_{k+l}(X_+ \wedge MG_l) \to \Omega_k^{\mathbf{G}}(X)$$

as follows: First we define a "collapse map" $c_l \colon \Omega_k^{\mathbf{G}}(X) \to \pi_{k+l}(X_+ \wedge MG_l)$ for each $l \geq l_0$ for some $l_0 \in \mathbb{N}$ large enough. Then we define an inverse map $d_l \colon \pi_{k+l}(X_+ \wedge MG_l) \to \Omega_k^{\mathbf{G}}(X)$ for all $l \geq l_1$ for some $l_1 \in \mathbb{N}$ large enough and show that c_l and d_l are inverses of one another for all $l \geq \max\{l_0, l_1\}$. Finally we show that for each $l \geq \max\{l_0, l_1\}$ the following diagram commutes.

$$\begin{tikzcd}
& \Omega_k^G(X) \arrow[dl, shift right, "c_l"'] \arrow[dr, shift left, "c_{l+1}"] & \\
\pi_{k+l}(X_+ \wedge MG_l) \arrow[ur, shift right, "d_l"'] \arrow[rr, "m_l"] & & \pi_{k+l+1}(X_+ \wedge MG_{l+1}) \arrow[ul, shift left, "d_{l+1}"]
\end{tikzcd}$$

Here m_l is induced by the suspension and $k_l\colon \Sigma MG_l \to MG_{l+1}$. Thus the maps d_l induce a map d from the colimit and d is an isomorphism.

<u>Construction of $c_l\colon \Omega_k^{\mathbf{G}}(X) \to \pi_{k+l}(X_+ \wedge MG_l)$:</u>

Let $[M^k, f, \gamma] \in \Omega_k^{\mathbf{G}}(X)$ be represented by a k-dimensional manifold M^k with an embedding $M^k \hookrightarrow S^{k+l}$ for some l large enough, a continuous map $f\colon M^k \to X$ and a G_l-structure $\gamma\colon \nu(M^k \hookrightarrow S^{k+l}) \to V_l$ on the normal bundle of M^k in S^{k+l}. Composing f with the bundle map $\nu(M^k \hookrightarrow S^{k+l}) \to M^k$ we obtain a map $\nu(M^k \hookrightarrow S^{k+l}) \to X$, which we are also going to call f. Then the structure on M^k can equivalently be characterised by a map

$$\nu(M^k \hookrightarrow S^{k+l}) \xrightarrow{(f,\ \gamma)} X \times V_l.$$

We want to use this structure to define a map

$$\alpha := c_l([M^k, f, \gamma])\colon S^{k+l} \to X_+ \wedge MG_l.$$

A *tubular neighbourhood* of a submanifold $j : M \hookrightarrow W$ is an embedding $J : \nu(M \hookrightarrow W) \hookrightarrow W$ of the normal bundle $\nu(M \hookrightarrow W)$ into W, restricting to the identity on the zero section: $J(0_x) = x$ for $x \in M$. By A.3 tubular neighbourhoods exist for boundaryless manifolds.

Let $J\colon \nu(M^k \hookrightarrow S^{k+l}) \to S^{k+l}$ be a tubular neighbourhood of M^k in S^{k+l}. Set $U := J(D(\nu(M^k \hookrightarrow S^{k+l})))$ to be the image of the disc bundle under J. Then U is a neighbourhood of M^k in S^{k+l}, diffeomorphic to the disc bundle. Consider the following composition of maps:

$$\begin{tikzcd}
S^{k+l} \arrow[r, "proj."] \arrow[d, dashed, "\alpha"] & (S^{k+l})/((S^{k+l} \setminus U)) \arrow[r, "\text{ind. by} J^{-1}_{|U}"] & D(\nu(M^k))/S(\nu(M^k)) \arrow[d, equal] \\
X_+ \wedge MG_l \arrow[r, equal] & \mathrm{Th}(X \times V_l) & \mathrm{Th}(\nu(M^k)) \arrow[l, "\mathrm{Th}(f,\ \gamma)"']
\end{tikzcd}$$

The above equalities are in fact isomorphism. While the vertical isomorphism is obvious, the horizontal isomorphism uses the following arguments: For two bundles $V_1 \to X_1$ and $V_2 \to X_2$ we have an isomorphism $\mathrm{Th}(V_1 \oplus V_2) \cong \mathrm{Th}(V_1) \wedge \mathrm{Th}(V_2)$ and considering X as a 0-dimensional vector bundle over itself we see that $\mathrm{Th}(X) = X_+$. Hence we obtain the required isomorphism:

$$\mathrm{Th}(X \times V_l) = \mathrm{Th}(X \oplus V_l) \cong \mathrm{Th}(X) \wedge \mathrm{Th}(V_l) = X_+ \wedge MG_l.$$

We have given an element $[\alpha] \in \pi_{k+l}(X_+ \wedge MG_l)$ but in order for our definition to be well-defined we need to check independence from the chosen representative (M^k, f, γ) of the bordism class $[M^k, f, \gamma]$:

Let $(M_0^k \hookrightarrow S^{k+l}, f_0, \gamma_0)$ and $(M_1^k \hookrightarrow S^{k+l}, f_1, \gamma_1)$ be two representatives of $[M^k, f, \gamma]$ and let $(W^{k+1} \hookrightarrow S^{k+l} \times I, F, \Gamma)$ be a $\mathbf{G}$-bordism between them. Since W^{k+1} restricts to $M_0^k \coprod M_1^k$ at the boundary, a tubular neighbourhood $\tilde{J}\colon \nu(W^{k+1}) \to S^{k+l} \times I$ can be chosen so that it equals the tubular neighbourhoods J_0 and J_1 of M_0^k and M_1^k, respectively, at the boundary ($\tilde{J}$ restricts to J_0 and J_1 in the fibers over ∂W^{k+1}). This implies that $\tilde{U} \cap (S^{k+l} \times \{0\}) = U_0$ and $\tilde{U} \cap (S^{k+l} \times \{1\}) = U_1$, where $U_0 = J_0(D(\nu(M_0^k)))$, $U_1 =$

$J_1(D(\nu(M_1^k)))$ and $\tilde{U} = \tilde{J}(D(\nu(W^{k+1})))$. We apply the Pontryagin-Thom construction to this map to obtain

$$\begin{tikzcd}
S^{k+l} \times I \arrow[r, "proj."] \arrow[d, dashed, "H"] & (S^{k+l} \times I)/(((S^{k+l} \times I) \setminus \tilde{U})) \arrow[r, "\text{ind. by } \tilde{J}^{-1}_{|\tilde{U}}"] & D(\nu(W^{k+1}))/S(\nu(W^{k+1})) \arrow[d, equal] \\
X_+ \wedge MG_l \arrow[r, equal] & \mathrm{Th}(X \times V_l) & \mathrm{Th}(\nu(W^{k+1} \hookrightarrow S^{k+n} \times I)) \arrow[l, "\mathrm{Th}(F,\, \Gamma)"']
\end{tikzcd}$$

The map H is a homotopy between the maps α_1 and α_2 obtained by using the respective representatives (M_0^k, f_0, γ_0) and (M_1^k, f_1, γ_1) in the above construction.

<u>Construction of the inverse maps $d_l\colon \pi_{k+l}(X_+ \wedge MG_l) \to \Omega_k^{\mathbf{G}}(X)$:</u>

Let $[\alpha\colon S^{k+l} \to (X_+ \wedge MG_l)] \in \pi_{k+l}(X_+ \wedge MG_l)$. Note that $X \times BG_l \hookrightarrow X_+ \times MG_l \to X_+ \wedge MG_l$, where the first map is the product of the inclusion $X \to X_+$ with the inclusion of BG_l as the zero section and the second map is the collapse map, is an embedding, because:

- $\{\infty\} \cap X = \emptyset$,
- $\{\infty\} \cap BG_l = \emptyset$,

where ∞ denotes the basepoint of X_+ and MG_l, respectively.

Our aim is now to find a map β homotopic to α such that $M^k := \beta^{-1}(X \times BG_l) \subset S^{k+l}$ is a smooth submanifold with a stable $\mathbf{G}$-structure on its normal bundle $\nu(M^k \hookrightarrow S^{k+l})$. The next lemma shows that there is a representative $\beta\colon S^{k+l} \to X_+ \wedge MG_l$ such that

1. For $X \times V_l = (X_+ \wedge MG_l) \setminus \{\infty\} \subset X_+ \wedge MG_l$ and $A := \beta^{-1}(X \times V_l)$,
$$\beta\colon A \to X \times V_l$$
is differentiable and transversal to the zero section $X \times BG_l \hookrightarrow X \times V_l$ of the fibre bundle
$$\begin{tikzcd}
X \times V_l \arrow[d, "\mathrm{id} \times Q_l"] \\
X \times BG_l
\end{tikzcd} .$$

2. For $M^k := \beta^{-1}(X \times BG_l)$, there is a tubular neighbourhood $J\colon \nu(M^k \hookrightarrow S^{k+l}) \xrightarrow{\cong} J(\nu(M^k)) := U \subset S^{k+l}$ such that
$$\beta(x) = \infty \Leftrightarrow x \notin U.$$

3. For the tubular neighbourhood U, the following map is a bundle map, i.e. a linear isomorphism in each fibre:
$$\begin{tikzcd}
\nu(M^k) \arrow[r, "J"] \arrow[d, "\nu"'] & U \arrow[r, "\beta"] & X \times V_l \arrow[d, "\mathrm{id} \times Q_l"] \\
M^k \arrow[rr, "\beta"] & & X \times BG_l
\end{tikzcd} .$$

By transversality, M^k is a smooth submanifold of S^{k+l}. $f\colon M^k \to X$ can be defined as $\mathrm{pr}_X \circ \beta$ and we get a bundle map $\gamma\colon \nu(M^k \hookrightarrow S^{k+l}) \to V_l$:

$$\begin{tikzcd}
\nu(W) \arrow[r, "\beta\circ J"] \arrow[d, "\nu"] & X \times V_l \arrow[r, "pr_2"] \arrow[d] & V_l \arrow[d, "p"] \\
W \arrow[r, "\beta"] & X \times BG_l \arrow[r, "\mathrm{pr}_2"] & BG_l
\end{tikzcd},$$

which defines a $\mathbf{G}$-structure for $\nu(M^k)$. Then $[M^k \subset S^{k+l}, f, \gamma] \in \Omega_k^{\mathbf{G}}(X)$ is a bordism class of an X-manifold. For the map to be well-defined, we still need to show that the construction is independent of the representative β of $[\alpha]$ satisfying the required properties.

Let $\delta\colon S^{n+l} \to X_+ \wedge MG_l$ be another representative satisfying the above properties 1, 2 and 3. Let $H\colon S^{k+l} \times I \to X_+ \wedge MG_l$ be a homotopy from β to δ. We are going to apply the three steps from Lemma 4.16 to this homotopy:

By step one we can assume that H is differentiable on $H^{-1}(X \times V_l) \subset S^{k+l} \times I$ and transversal to $X \times BG_l$. By step two we can assume that H maps a tubular neighbourhood $\tilde{U} \xrightarrow{\tilde{J}^{-1}} \nu(H^{-1}(X \times BG_l) \hookrightarrow S^{k+l} \times I)$ to $X \times V_l$ and its complement to ∞. By a step analogous to step three we extend the bundle maps

$$\begin{tikzcd}
\nu(\beta^{-1}(X \times BG_l) \hookrightarrow S^{k+l}) \arrow[r] \arrow[d, "\nu"] & X \times V_l \arrow[d] \\
\beta^{-1}(X \times BG_l) \arrow[r, "\beta"] & X \times BG_l
\end{tikzcd}$$

and

$$\begin{tikzcd}
\nu(\delta^{-1}(X \times BG_l) \hookrightarrow S^{k+l}) \arrow[r] \arrow[d, "\nu"] & X \times V_l \arrow[d] \\
\delta^{-1}(X \times BG_l) \arrow[r, "\delta"] & X \times BG_l
\end{tikzcd}$$

to

$$\begin{tikzcd}
\nu(H^{-1}(X \times BG_l) \hookrightarrow S^{k+l} \times I) \arrow[r] \arrow[d] & X \times V_l \arrow[d] \\
H^{-1}(X \times BG_l) \arrow[r, "H"] & X \times BG_l
\end{tikzcd}$$

with induced $\mathbf{G}$-structure $\nu(H^{-1}(X \times BG_l)) \to X \times V_l \xrightarrow{proj.} V_l$ and map $H^{-1}(X \times BG_l) \xrightarrow{H} X \times BG_l \xrightarrow{\pi_1} X$. Then $H^{-1}(X \times BG_l)$ is a $\mathbf{G}$-bordism between $(\beta^{-1}(X \times BG_l), \mathrm{pr}_1 \circ \beta)$ and $(\delta^{-1}(X \times BG_l), \mathrm{pr}_1 \circ \delta)$. We have thus shown that $d_l\colon \pi_{k+l}(X \wedge MG_l) \to \Omega_k^{\mathbf{G}}(X)$ is well-defined.

<u>Commutativity of the d_l's:</u>

$$\begin{tikzcd}
\pi_{n+l}(X_+ \wedge MG_l) \arrow[rr, "m_l"] \arrow[dr, "d_l"'] & & \pi_{n+l+1}(X_+ \wedge MG_{l+1}) \arrow[dl, "d_{l+1}"] \\
& \Omega_n^{\mathbf{G}}(X) &
\end{tikzcd}.$$

The upper homomorphism takes a class $[\alpha\colon S^{k+l} \to X_+ \wedge MG_l]$ to the class

$$[m_l([\alpha]) : S^{k+l+1} = \Sigma(S^{k+l}) \xrightarrow{\Sigma\alpha} \Sigma(X_+ \wedge MG_l) = X_+ \wedge \Sigma(MG_l) \xrightarrow{id\wedge k_l} X_+ \wedge MG_{l+1}].$$

A generalisation of Theorem 3.12 shows that $d_{l+1}([m_l([\alpha])]$ is the same manifold M^k (as obtained as $d_l([\alpha])$), embedded into the equator S^{k+l} of S^{k+l+1}, with framing the direct sum of the old framing and the trivial 1-dimensional framing. $(M^k \subset S^{k+l}, f, \gamma)$ and $(M^k \subset S^{k+l} \subset S^{k+l+1}, f, \gamma \oplus \epsilon)$ are clearly bordant.

By properties 1, 2 and 3 above, d_l is constructed so that $d_l \circ c_l = \mathrm{id}_{\Omega_k^{\mathbf{G}}(X)}$. By the definition of c_l, we have $c_l \circ d_l = \mathrm{id}_{\pi_{k+l}(X_+\wedge MG_l)}$.

□

Lemma 4.16. *Every element $[\alpha] \in \pi_{n+l}(X_+ \wedge MG_l) = [S^{n+l},\ X_+ \wedge MG_l]_0$ has a representative $\beta\colon S^{n+l} \to X_+ \wedge MG_l$ such that the three conditions in Theorem 4.15 are fulfilled.*

Proof. Let $[\alpha] \in \pi_{n+l}(X_+ \wedge MG_l)$ be represented by $\alpha\colon S^{n+l} \to X_+ \wedge MG_l$. Set $A \coloneqq \alpha^{-1}(X \times V_l)$. We will homotope α in three steps to obtain a homotopic map β fulfilling the three conditions.

Claim 1: There is a map $\alpha_1\colon S^{n+l} \to X_+ \wedge MG_l$ (pointed) homotopic to α such that $\alpha_1^{-1}(X \times V_l) = A$ and $\alpha_1\colon A \to X \times V_l$ is differentiable and transversal to the zero section $X \times BG_l$.

Proof: By Theorem A.2 and Thom's Transversality Theorem [A.10], there is a homotopy $H\colon A \times I \to X \times V_l$ such that $H_0 = \alpha$ and H_1 is differentiable. By Theorem A.11, there is a homotopy $K\colon A \times I \to X \times V_l$ such that $K_0 = H_1$ and K_1 is differentiable and transversal to $X \times BG_l \subset X \times V_l$. H and K can be extended to homotopies $H, K\colon S^{n+l} \to X_+ \wedge MG_l$ by setting $H(x,t) = \infty = K(x,t)$ for $x \notin A$, because H and K are proper maps, i.e. preimages of compact sets are compact. Set $\alpha_1 := K_1$.

Claim 2: Set $W := \alpha_1^{-1}(X \times BG_l)$ and $j\colon W \hookrightarrow S^{n+l}$. Let $U_\epsilon \subset A \subset S^{n+l}$ be a tubular neighbourhood of W, i.e. there is a diffeomorphism $J\colon \nu(W) \to U_\epsilon$. Then there is a map $\alpha_2\colon S^{n+l} \to X_+ \wedge MG_l$ (pointed) homotopic to α_1 such that

- $U_\epsilon = \alpha_2^{-1}(X \times V_l)$, i.e. $\alpha_2(x) = \infty \Leftrightarrow x \notin U_\epsilon$, and
- $\alpha_2\colon U_\epsilon \to X \times V_l$ is differentiable and transversal to the zero section $X \times BG_l$.

Proof: Let $\lambda\colon S^{n+l} \to [0,1]$ be a differentiable function with

$$\lambda^{-1}(0) = U_{\epsilon/2} \qquad \text{and} \qquad \lambda^{-1}([0,1)) = U_\epsilon.$$

Define $H\colon S^{n+l} \times I \to X_+ \wedge MG_l$ as

$$H(X,t) = \begin{cases} \frac{1}{1-t\lambda(x)} \cdot \alpha_1(x) & \text{for } x \in A \text{ and } t < 1, \text{ or } x \in U_\epsilon \text{ and } t = 1, \\ \infty & \text{else,} \end{cases}$$

where $\cdot$ denotes scalar multiplication in each fibre of $X \times V_l$.

Then H is a homotopy with

$$H(x,0) = \alpha_1(x)$$

for all $x \in S^{n+l}$ and

$$H(x,1) = \begin{cases} \frac{1}{1-\lambda(x)} \cdot \alpha_1(x) & \text{for } x \in U_\epsilon, \\ \infty & \text{else.} \end{cases}$$

Set $\alpha_2 := H_1$. Then:

$$x \notin U_\epsilon \Rightarrow \alpha_2(x) = \infty, \text{ and } x \in U_\epsilon \Rightarrow \alpha_2(x) \in X \times V_l.$$

The latter is true because $\alpha_1(x)$ is just modified by a scalar multiplication within a fibre. So $U_\epsilon = \alpha_2^{-1}(X \times V_l)$ is fulfilled. By construction, α_2 is differentiable. Since $W = \alpha_1^{-1}(X \times BG_l) \subset U_{\epsilon/2}$ and $\alpha_{2|U_{\epsilon/2}} = \alpha_{1|U_{\epsilon/2}}$, the map α_2 is also transversal to $X \times BG_l$.

Claim 3: There is a map $\beta\colon S^{n+l} \to X_+ \wedge MG_l$ homotopic to α_2 such that

- $U_\epsilon = \beta^{-1}(X \times V_l)$ (as before),
- $\beta\colon U_\epsilon \to X \times V_l$ is differentiable and transversal to the zero section $X \times BG_l$ (as before) and
- $$\begin{tikzcd} \nu(W \hookrightarrow S^{n+l}) \arrow[r, "J"] \arrow[d, "\nu"] & U_\epsilon \arrow[r, "\beta"] & X \times V_l \arrow[d, "\mathrm{id} \times Q_l"] \\ W \arrow[rr, "\beta"] & & X \times BG_l \end{tikzcd}$$

  is a differentiable bundle map.

Proof: Consider the composition

$$h\colon \nu(W) \xrightarrow{J} U_\epsilon \xrightarrow{\alpha_2} X \times V_l.$$

Since α_2 is differentiable and transversal to the zero section $X \times BG_l$, and J is a diffeomorphism, h is also differentiable and transversal to $X \times BG_l$. Define a homotopy $H\colon \nu(W) \times [0,1] \to X \times V_l$ by

$$H_t(x) := \frac{1}{t} \cdot h(t \cdot x) \text{ for } t > 0,$$

and extend it continuously to $\nu(W) \times [0,1]$. Here $\cdot$ denotes scalar multiplication in each fibre.

Locally h is of the form

$$U \times \mathbb{R}^l \to V \times \mathbb{R}^l,\ (u,v) \mapsto (h_1(u,v), h_2(u,v))$$

for $U \times \mathbb{R}^l \to \nu^{-1}(U),\ U \subset W$ a chart for ν and $V \times \mathbb{R}^l \to Q_l^{-1}(V),\ V \subset X \times BG_l$ a chart for Q_l. So in terms of these charts, H is of the from

$$H_t(x) = (h_1(u, t \cdot v), \frac{1}{t} \cdot h_2(u, t \cdot v)) \text{ for } t > 0.$$

Restricted to one fibre h is of the from $h_u\colon \mathbb{R}^l \to \mathbb{R}^l,\ h_u(x) = h_2(u,x)$ with

$$\lim_{t\to 0}(\frac{1}{t} \cdot h_2(u,tv)) = \lim_{t\to 0}(\frac{h_u(tv)}{t}) = D_0 h_u \in \mathbb{R}^{l\times l}.$$

Then $H_0(u,v) = \lim_{t\to 0}(h_1(u,tv), \frac{1}{t}\cdot h_2(u,tv)) = (h_1(u,0), D_0h_u)$. So H_0 restricted to the fibre over $u \in U$ is the linear map D_0h_u. Because h is transversal to $X \times BG_l$, D_0h_u is surjective onto the l-dimensional normal space of $X \times BG_l$ in $X \times V_l$ at $Q_l^{-1}(u)$. Thus H_0 is a linear isomorphism in each fibre.

Set

$$K\colon U_\epsilon \times I \xrightarrow{J^{-1}\times \mathrm{id}} \nu(W) \times I \xrightarrow{H} X \times V_l$$

and extend to a homotopy

$$K\colon S^{n+l} \times I \to \nu(W) \times I \to X_+ \wedge MG_l.$$

by sending any $x \notin U_\epsilon$ to ∞. Again this is possible because K is proper.

Then $\beta := K_1$ has the desired properties of the claim respectively of the lemma. □

4.6. Some Bordism Groups

In the following I am going to state some facts about the bordism groups corresponding to the empty structure $\mathbf{O} := \{O(l), \mathrm{id}, O(l) \hookrightarrow O(l+1)\}$ called "unoriented bordism groups", and the "oriented bordism groups" corresponding to the orientation preserving structure $\mathbf{SO} := \{SO(l), SO(l) \hookrightarrow O(l), SO(l) \hookrightarrow SO(l+1)\}$. These results are taken from [3] and can be found there in more detail.

We call the groups $\Omega_k^{\mathbf{G}} := \Omega_k^{\mathbf{G}}(pt)$ the *coefficients* of the generalised homology theory. Note that $pt_+ \wedge M = M$ for a manifold M, so by the previous Theorem 4.15

$$\Omega_k^{\mathbf{G}} \cong \lim_{l\to\infty} \pi_{n+l}(MG_l).$$

Here is a summary of some known results about oriented bordism groups:

k	0	1	2	3	4
$\Omega_k^{\mathbf{SO}}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

While it is more complicated to characterise the oriented bordism groups for general manifolds, there is an easy observation for unoriented bordism groups. As indicated in the introduction, any element in $\Omega_k^{\mathbf{O}}(X)$ is of order two for any $k \in \mathbb{N}$ and any manifold X. Therefore all oriented bordism groups $\Omega_k^{\mathbf{O}}(X)$ consist only of elements of order two. A theorem of Thom which can be found in [3, Ch. 10.10] gives all unoriented bordism groups of a point.

A. Appendix

The definitions and theorems listed in this appendix can be found in [1], [3] and [8].

A.1. Differential Topology

Theorem A.1. *[Whitney embedding theorem]*
Let $\epsilon\colon M^m \to \mathbb{R}$ be a strictly positive map and $f\colon M^m \to \mathbb{R}^p$, $p > 2n$, a map that is an embedding in a neighbourhood of the closed subset $A \subset M^m$. Then there is an ϵ-approximation $g : M^m \to \mathbb{R}^p$ of f such that $g_{|A} = f_| A$ and g is an embedding of M^m with $g(M^m) \subset \mathbb{R}^p$ closed. g is called ϵ-approximation of f if the distance of $f(x)$ and $g(x)$ is less than $\epsilon(x)$ for all $x \in M^m$ (w.r.t. a given metric on $\mathbb{R}^p$). f only needs to be continuous for this, as shows the next theorem.

Theorem A.2. *Let $f\colon M \to N$ be a continuous map, differentiable on the closed subset $A \subset M$. Let $\epsilon\colon M \to \mathbb{R}$ be strictly positive and suppose N carries a metric. Then there is a differentiable ϵ-approximation $g\colon M \to N$ of f with $g_{|A} = f_{|A}$*

Definition and Lemma A.3. *[Tubular neighbourhood]*
Let M be a boundaryless manifold and $j\colon N \hookrightarrow M$ an embedding of a submanifold. Then there is an embedding $J : \nu(N) \to M$ extending j on the zero section $N \subset \nu(N)$ and mapping $\nu(N)$ diffeomorphically onto an open neighbourhood $U \subset M$ of $J(N)$. This neighbourhood U is called tubular neighbourhood *of N in M.*

Definition A.4. [Regular value]
Let $f\colon M \to N$ be a smooth map of smooth manifolds. $x \in M$ is called a *regular point* of f if the differential $D_x f$ is nonsingular. A point $y \in N$ is called a *regular value* if $f^{-1}(y)$ contains only regular points.

Theorem A.5. *[Theorem of Sard] [8]*
Let $f\colon U \to \mathbb{R}^n$ be a smooth map with $U \subset \mathbb{R}^m$ open and set

$$C := \{x \in U \mid \operatorname{rank} D_x f < n\}.$$

Then the image $f(X) \subset \mathbb{R}^n$ has Lebesgue measure zero.

Corollary A.6. *[Corollary by Brown]*
Let $f\colon M^m \to N^n$, be a smooth map of smooth manifolds, $m \geq n$. The set of regular values of f is everywhere dense in N.

Lemma A.7. *Let $f\colon M^m \to N^n$ be a map of manifolds with $m \geq n$ and $z \in N$ a regular value of f. Then the set $f^{-1}(y) \subset M$ is a submanifold of M of dimension $m - n$.*

A.2. Transversality

In the following we consider all maps between smooth manifolds to be smooth if not otherwise stated.

Let $f\colon M \to N$ be a map between manifolds M and N, let $U \subset N$ be a submanifold of N. We want $f^{-1}(U) \subset M$ to be a submanifold of M, which unfortunately is not always the case, nevertheless it is in almost all cases. This is a generalisatoin of the Sard Theorem.

Definition A.8. Let $U^k, V^l \subset N^n$ be two submanifolds of the manifold N. We say that U and V intersect *transversally* in $p \in U \cap V$ if $T_pU + T_pV = T_pN$. The submanifolds U and V are *transverse* if they intersect transversally in every point of intersection. If U and V do not intersect, they are said to be *vacuously transverse.*

Let $f\colon M^m \to N^n$ be a map of manifolds and $U^{n-k} \subset N$ a submanifold. The map f is *transverse* to U in $x \in M$ if

$$f(x) \in U \Rightarrow T_{f(x)}U + T_xf(T_xM) = T_{f(x)}N,$$

i.e. T_xM should be mapped surjectively onto $T_{f(x)}/T_{f(x)}U$. The map f is called *transverse* to U if it is transverse to U in every point $x \in M$. Note that if $U = \{pt\}$ is a point transverse to f, it is a *regular value* of f.

Two maps $f, g\colon M^m \to N^n$ are called *transverse* in $x \in M$ if

$$f(x) = g(x) \Rightarrow T_xf(T_xM) + T_xg(T_xM) = T_f(x)N,$$

i.e. the images of the tangent space of M under the differentials of f and g generate the tangent space of N. The maps f and g are called *transverse* if they are transverse in every point $x \in M$.

Theorem A.9. *If $f\colon M^m \to N^n$ is a map of manifolds transverse to the submanifold $U^{n-k} \subset N$, then $f^{-1}(U)$ is a submanifold of M of dimension $m - k$.*

Theorem A.10. *[Thom's Transverality Theorem]*
Let $f\colon M^m \to N^n$ be a map of manifolds and $U \subset N$ a closed submanifold. Let $A \subset M$ be closed with f transversal to U in every point $x \in A$. Let $\delta\colon M \to \mathbb{R}$ be strictly positive and N a manifold carrying a metric. Then there is a δ-approximation $g\colon M \to N$ of f with $g_{|A} = f_{|A}$, transversal to U.

In particular any continuous map can be homotoped to a transversal map:

Theorem A.11. *Let $f\colon M^m \to N^n$ be a continuous map of smooth manifolds, let N carry a metric. Then for every strictly positive map $\epsilon\colon M \to \mathbb{R}$ there is a strictly positive map $\delta\colon M \to \mathbb{R}$ such that:*
If g is a δ-approximation of f, then g is homotopic to f by a homotopy $F(x,t)$ with

- *$F(x,t) = f(x)$, if $g(x) = f(x)$,*
- *$F(x,t)$ is an ϵ-approximation of f for each $t \in [0,1]$.*

A.3. Fibre Bundles

Definition A.12. A *fibre bundle* is given by a quadruple (p, E, B, F) where E, B and F are topological spaces and $p\colon E \to B$ is a map of topological spaces together with a collection of homeomorphisms $\varphi\colon U \times F \to p^{-1}(U)$ for open sets U in B called *charts over U* satisfying the following conditions:

1. For each $b \in B$ there is a neighbourhood U with a chart $\varphi\colon p^{-1}(U) \to U$.
2. If φ is a chart over $U \subset B$ and $V \subset U$ is open, then $\varphi_{|V}$ is a chart over V.

3. p is locally trivial, i.e. for each chart $\varphi\colon p^{-1}(U) \to U$, the following diagram commutes:

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi"] \arrow[dr, "pr_U"'] & & p^{-1}(U) \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

4. The collection of charts is maximal among those satisfying the previous three conditions.

E is called *the total space*, B *the base space* and F *the fibre*.

Definition A.13. Let G be a topological group acting on a topological space F. Then a *fibre bundle with structure group* G is a fibre bundle (p, E, B, F) as above such that additionaly:

3.b For any two charts $\varphi,\ \varphi'$ over U, there exists a continuous function $\theta_{\varphi,\varphi'}\colon U \to G$ called *transition function* or *change of charts* such that

$$\varphi'(u, f) = \varphi(u, \theta_{\varphi,\varphi'}(u) \cdot f)$$

for all $u \in U,\ f \in F$.

holds.

Lemma A.14. *Given spaces B and F, a group G acting on F from the left and a collection of transition functions $T = (U_\alpha, \theta_\alpha\colon U_\alpha \to G)$ such that:*

1. *The U_α cover B,*
2. *$(U, \theta) \in T$ and $W \subset U \Rightarrow (W, \theta_{|W}) \in T$,*
3. *$(U, \theta_1),\ (U, \theta_2) \in T \Rightarrow (U, \theta_1 \cdot \theta_2) \in T$, where $\cdot$ denotes pointwise multiplication,*
4. *T is maximal with respect to these three properties.*

Then there exists a fibre bundle $p\colon E \to B$ with structure group G, fibre F and transition functions θ_α unique up to fibre isomorphism.

Definition A.15. For a topological group G a *principal G-bundle over B* is a fibre bundle $p\colon P \to B$ with fibre $F = G$ and structure group G acting on itself by left translation:

$$G \to \mathrm{Homeo}(G),\ g \mapsto (x \mapsto g \cdot x).$$

Proposition A.16. *If $p : P \to B$ is a principal G-bundle, then G acts freely on P from the right.*

Changing the fibre: By Lemma A.14, the transition functions determine a bundle. So given a fibre bundle $p\colon E \to B$ with fibre F and structure group G, we can change p to another fibre bundle $p'\colon E' \to B$ with the same transition functions, the same structure group G and a new fibre F', under the conditions that G acts on F' from the left. This can especially be done to construct principal bundles from fibre bundles:

Given a fibre bundle

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd},$$

we can always change the fibre from F to $F' := G$ since G acts on itself by left translation. The resulting bundle

$$\begin{tikzcd} G \arrow[r] & P(E) \arrow[d, "p"] \\ & B \end{tikzcd}$$

is then principal, called the *principal G-bundle underlying the fibre bundle $p\colon E \to B$ with structure group G*. The construction also works in the other direction: Given a principal G-bundle and a space F acted upon by G, we can construct an associated fibre bundle with fibre F and the same transition functions as the principal bundle. An alternative construction is given by the *Borel construction*:

Definition and Lemma A.17. *Given a principal G-bundle $p\colon E \to B$ and a space F acted upon by G, set*

$$P \times_G F := (P \times F)_{/\sim}, \text{ where } (x, f) \sim (xg, g^{-1}f) \text{ for all } x \in P, f \in F, g \in G$$

and

$$q\colon P \times_G F \to B,\ [x, f] \mapsto p(x).$$

Then

$$\begin{tikzcd} F \arrow[r] & P \times_G F \arrow[d, "q"] \\ & B \end{tikzcd}$$

is a fibre bundle over B with structure group G and the same transition functions as p.

Definition and Theorem A.18. *For every topological group G there exists a principal G-bundle*

$$EG \to BG$$

where EG is a contractible space. This bundle is called the universal principal G-bundle. *The space BG is called* the classifying space *for G and has the following property:*
The map

$$\Phi\colon Maps(B, BG) \to \{Principal\ G\text{-bundles over } B\}$$

defined by pulling back the universal principal bundle $EG \to BG$ along the map $c\colon B \to BG$ (so $\Phi(c) = c^*(EG)$) induces a bijection from the homotopy set $[B, BG]$ to the set of isomorphism classes of principal G-bundles over B, when B is a paracompact space.
For a principal G-bundle $P \to B$, the map $c\colon B \to BG$ with $P = c^*(EG)$ is called the classifying map *for $P \to B$.*

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Erklärung

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Karlsruhe, den 29.01.2018