# I<sup>1</sup>-Measure equivalence rigidity of hyperbolic lattices

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Banff, March 2008

## *IP*-Measure equivalence

## Measure equivalence with $I^p$ -condition

A ME-coupling  $(\Omega, \mu)$  of  $\Gamma$  and  $\Lambda$  is a measure space with a  $\mu$ -preserving action of  $\Gamma \times \Lambda$  such that  $\Gamma$ ,  $\Lambda$  both have a  $\mu$ -finite fundamental domain. If  $\Gamma$  and  $\Lambda$  admit a ME-coupling with  $I^p$ -integrable cocycles w.r.t. some fundamental domains then we call them  $I^p$ -measure equivalent.

## *I<sup>p</sup>*-coycles

A measurable cocycle  $\alpha: \Gamma \times (X, \mu) \to \Lambda$  is  $I^p$ -integrable if for every  $\gamma \in \Gamma$ 

$$\int_X I(\alpha(\gamma,x))^p d\mu(x) < \infty,$$

where  $I: \Lambda \to \mathbb{N}$  is the length function for some word metric on  $\Lambda$ .

- $I^p$ -ME interpolates between  $p = \infty$  (  $\Rightarrow$  QI) and p = 0 (=ME).
- *IP*-ME is an equivalence relation on groups.



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# Rigidity result for hyperbolic lattices

## Theorem (informal)

Let  $\Gamma$  be a lattice in  $G = \text{Isom}(\mathbb{H}^n)$ ,  $n \geq 3$ . Then any  $l^1$ -ME-coupling of  $\Gamma$  with another group basically comes from the standard example of lattices in G or atomic couplings of commensurable groups.

- The standard coupling of hyperbolic lattices is  $l^1$ -integrable.
- A corresponding rigidity result for orbit equivalence (OE) can be formulated.
- Analogous rigidity results (without any I<sup>1</sup>-integrability condition) for lattices in higher rank Lie groups hold true [Furman, 2000].
- Lack of rigidity for n = 2:  $\mathbb{Z}^2 * \mathbb{Z}^2$  OE to  $\mathbb{Z} * \mathbb{Z}$ .

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# Precise rigidity result – ME-version

## Theorem (Bader-Furman-S.)

Let  $\Gamma$  be a lattice in  $G = \text{Isom}(\mathbb{H}^n)$ ,  $n \geq 3$ . Let  $(\Omega, \mu)$  be an ergodic,  $l^1$ -integrable ME-coupling with another group  $\Lambda$ .

Then the following holds:

- a) There exists a homomorphism  $\rho:\Lambda\to G$  with finite kernel and image being a lattice in G.
- b) There exists a  $\Gamma \times \Lambda$ -equivariant measurable map  $\phi : \Omega \to G$ ; the push-forward measure  $\phi_*\mu$  is the Haar measure corresponding either
  - i) to G,
  - ii) or to its index two subgroup  $G^0 = \text{Isom}_+(\mathbb{H}^n)$ ,
  - iii) or to a lattice  $\Gamma' < G$ .

In the latter case,  $\Gamma'$  contains  $\Gamma$  and  $\rho(\Lambda)$  as subgroups of finite index.

## Theorem (Mostow rigidity – Lie-theoretic version)

Any isomorphism  $\Gamma \to \Lambda$  of lattices in  $G = \text{Isom}(\mathbb{H}^n)$ ,  $n \geq 3$ , extends to an automorphism of G.

# Theorem (Mostow rigidity – topological version)

Let M and N be closed hyperbolic n-dimensional manifolds. Then any homotopy equivalence  $M \to N$  is homotopic to an isometry.

topological version  $\Rightarrow$  Lie-theoretic version:



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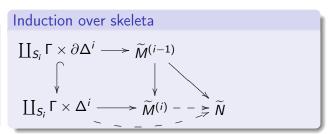
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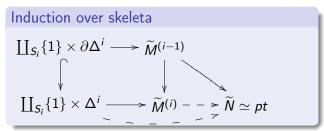
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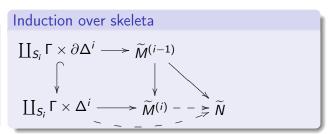
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# Thurston's proof of (topological) Mostow rigidity

#### Proof for closed manifolds

- Step 1)  $f: M \xrightarrow{\simeq} N \Rightarrow ||M|| = ||N|| \Rightarrow vol(M) = vol(N)$  [Gromov-Thurston].
- Step 2)  $\tilde{f}: \mathbb{H}^n \to \mathbb{H}^n$  is a quasi-isometry, thus induces a homeomorphism  $\partial_\infty \tilde{f}: \partial_\infty \mathbb{H}^n \xrightarrow{\cong} \partial_\infty \mathbb{H}^n$ .
- Step 3) Regular, ideal *n*-simplices are exactly the geodesic *n*-simplices with maximal volume [Haagerup-Munkholm].  $\partial_{\infty} \tilde{f}$  preserves regular, ideal simplices.
- Step 4) Hyperbolic geometry:  $\partial_{\infty} \tilde{f}$  induced by an isometry.

#### Modification for finite volume manifolds

Only from volume considerations, Thurston constructs a measurable  $\partial_{\infty} \tilde{f}$  that preserves regular, ideal *n*-simplices almost everywhere.



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# Reduction of main theorem to cocycle Mostow rigidity I

## Theorem (adapted from Furman's earlier work)

Let  $\Gamma$  be a lattice in  $G = \text{Isom}(\mathbb{H}^n)$ , and  $\Lambda$  be an arbitrary group ME to  $\Gamma$  via the coupling  $(\Omega, m)$ . Let  $(\Sigma, n) = (\Omega, m) \times_{\Lambda} (\Omega^{\operatorname{op}}, m)$  be the corresponding self-coupling of  $\Gamma$ . Assume that there exists a measurable  $\Gamma \times \Gamma$ -equivariant map  $\Phi : \Sigma \to G$  ("untwisting map"), i.e. n-a.e.

$$\Phi([\gamma x, \gamma' y]) = \gamma \Phi([x, y]) \gamma'^{-1} \qquad (\gamma, \gamma' \in \Gamma).$$

Then there exist measurable maps  $f:\Omega\to G$  and a homomorphism  $\rho:\Lambda\to G$  so that

$$f((\gamma, \lambda)x) = \gamma f(x)\rho(\lambda)^{-1}.$$

Then elementary observations (for lattice image) and an application of Ratner's theorems (for identifying  $\Phi_* n$ ) eventually yield the main theorem.

... How do we get the untwisting map?

# Reduction of main theorem to cocycle Mostow rigidity II

#### Setting

- Let X ⊂  $\Sigma$  be a common fundamental domain of both copies of Γ, and  $\alpha$  : Γ × X → X be the corresponding OE-cocycle.
- We may assume that  $\Gamma \subset \text{Isom}(\mathbb{H}^n)$  is co-compact. Let  $M = \Gamma \backslash \mathbb{H}^n$ .

#### Proof of main theorem - outline

- Step 1) Extend  $\alpha: X \to \operatorname{map}(\Gamma, \Gamma)$  to a  $\alpha$ -equivariant, measurable map  $\psi: X \to \operatorname{map}(\widetilde{M}, \widetilde{M})$ .
- Step 2) Show that  $\psi$  induces a measurable,  $\alpha$ -equivariant map  $\partial_{\infty}\psi:X\to\mathcal{M}(\partial\mathbb{H}^n,\partial\mathbb{H}^n)$  that preserves regular, ideal n-simplices.
- Step 3) Hyperbolic geometry  $\Rightarrow \partial_{\infty} \psi$  comes from of a  $\alpha$ -equivariant map  $\phi: X \to \mathsf{Isom}(\widetilde{M}) = G$  (cocycle Mostow rigidity).
- Step 4)  $\phi$  is a coboundary for  $\alpha$ ; thus we can also untwist  $\Sigma$ .

# A crucial step in the proof – controlling volume

#### Lemma

For any geodesic simplex  $\sigma$  with  $vol(\sigma) \approx v_{max}$  we have

$$\int_X \int_{\Gamma \backslash G} \operatorname{vol}(\psi_{\mathsf{X}}(g\sigma)) d\mu_{\mathsf{X}}(\mathsf{X}) d\mu_{\Gamma \backslash G}(g) \approx \operatorname{vol}(\sigma).$$

### Volume and degree 1 maps

Let  $f: M \to M$  be a degree 1 map. Let  $c = \sum a_i \sigma_i$  be an *n*-cycle. Then

$$\sum a_i\operatorname{\mathsf{vol}}^{\operatorname{or}}(\sigma_i) = \sum a_i\operatorname{\mathsf{vol}}^{\operatorname{or}}(f(\sigma)).$$

- Find suitable homology theories for our situation.
- Show that  $\psi: X \to \operatorname{map}(\widetilde{M}, \widetilde{M})$  is of degree 1.
- View left side of lemma as the evaluation of a homology class at the volume form.

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# l<sup>1</sup>-homology and induced maps

#### Remarks

- $L^1(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C^{\mathrm{geo}}_*(\widetilde{M}) = \bigoplus_F L^1(X;\mathbb{Z})$
- Vertical maps are inclusions of orbits.
- $C_0(\alpha)(1 \otimes \gamma) = \sum \chi_{X_i} \otimes \gamma_i$  where  $\alpha(x, \gamma) = \gamma_i$  constant on  $x \in X_i$ .

# A new deformation-rigidity phenomenon

## Integrality, Poincare duality, simplicial volume

We have by Poincare duality and ergodicity

$$H_n(L^1(X;\mathbb{Z})\otimes_{\mathbb{Z}\Gamma}C_*^{\mathrm{geo}}(\widetilde{M}))\ \cong H^0(\widetilde{M};L^1(X;\mathbb{Z}))=L^1(X;\mathbb{Z})^\Gamma\ \cong \mathbb{Z}.$$

Since the simplicial volume of M is > 0 every Cauchy sequence of cycles in  $L^1(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n^{\mathrm{geo}}(\widetilde{M})$  is eventually constant!

## Sobolev homology and I<sup>1</sup>-condition

Under  $I^1$ -integrability, we show that  $H_n(\phi)$  already lands in

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