A new tool for computing $L^2$-Betti numbers of groups

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\textbf{Definition for Riemannian manifolds (Atiyah)}

Let $\tilde{M} \to M$ be the universal covering, and let $\mathcal{F} \subset \tilde{M}$ be a $\pi_1(M)$-fundamental domain. Then define

$$b^{(2)}_p(\tilde{M} : \pi_1(M)) = \int_{\mathcal{F}} \text{tr}_C e^{-t\Delta_p}(x, x) d\text{vol}(x).$$

\textbf{Simplicial definition (Dodziuk)}

For a (finite) simplicial complex $K$ with $\Gamma = \pi_1(K)$, define $b^{(2)}_p(\tilde{K} : \Gamma)$ as the Murray-von Neumann dimension of the Hilbert $\Gamma$-module

$$\bar{H}^p(\tilde{K}, l^2(\Gamma)).$$

For a group $\Gamma$ we set $b^{(2)}_p(\Gamma) = b^{(2)}_p(\mathcal{E}\Gamma : \Gamma)$. 
Definition for Riemannian manifolds (Atiyah)

Let $\tilde{M} \to M$ be the universal covering, and let $\mathcal{F} \subset \tilde{M}$ be a $\pi_1(M)$-fundamental domain. Then define

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For a group $\Gamma$ we set $b_{p}^{(2)}(\Gamma) = b_{p}^{(2)}(E\Gamma : \Gamma)$. 
Lück and Farber defined different algebraic definitions to extend the $L^2$-Betti numbers to arbitrary $\Gamma$-spaces and groups. Lück’s machinery allows the use of standard spectral sequences to compute $L^2$-Betti numbers.

Gaboriau defined $b_p^{(2)}(R)$ for a measured equivalence relation $R$. Later on, generalization to discrete measured groupoids.

Example: Orbit equivalence relation of $\Gamma \curvearrowright (X, \mu)$, where $(X, \mu)$ is a probability space. In this case, $b_p^{(2)}(R) = b_p^{(2)}(\Gamma)$. Any infinite amenable group is orbit equivalent to $\mathbb{Z}$.

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Connes-Shlyakhtenko defined $b_{\rho}^{(2)}(A)$ for an arbitrary finite von Neumann algebra $A$. 
von Neumann algebras and dimension theory

Finite von Neumann algebras

- Finite von Neumann algebras are weakly closed $\ast$-subalgebras of some $\mathcal{B}(H)$ with a finite trace $\text{tr}$ which has the trace property $\text{tr}(ab) = \text{tr}(ba)$.
- $L^\infty(X, \mu)$ with $\text{tr}(f) = \int_X f d\mu$
- group von Neumann algebra: $L(\Gamma) = \mathcal{B}(l^2\Gamma)^\Gamma$ ($\Gamma$-equivariant bounded operators) with trace $\text{tr}(a) = \langle a(1), 1 \rangle$.

Dimension function for arbitrary modules (Lück)

There exists an additive dimension function

$$\dim_A : \{A\text{-modules}\} \to [0, \infty]$$

such that if $p \in A$ is a projection then $\dim_A(\mathcal{A}p) = \text{tr}_A(p)$.
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von Neumann algebras and dimension theory
Discrete measured groupoids

Please don’t! Discrete measured groupoids can be used effectively to compute $L^2$-Betti numbers of groups.
Discrete measured groupoids

Examples
- **translation groupoid** $X \ltimes \Gamma$ of a $\mu$-preserving action $\Gamma \rhd (X, \mu)$. If the action is free then $X \ltimes \Gamma$ is the orbit equivalence relation.
- **holonomy groupoids** of measured foliations (restricted to a transversal)

Groupoid ring and von Neumann algebra of a groupoid $\mathcal{G}$
- **Groupoid ring** $\mathbb{C}\mathcal{G}$ consists of finitely supported Borel functions $\mathcal{G} \to \mathbb{C}$ equipped with a convolution product. $L^\infty(\mathcal{G}^0)$ is $\mathbb{C}\mathcal{G}$-module.
- $\mathbb{C}\mathcal{G}$ carries a trace, and its von Neumann algebra completion is denoted by $L(\mathcal{G})$.
- If $\mathcal{G}$ is the orbit equivalence relation of $\Gamma \rhd (X, \mu)$ then $\mathbb{C}\mathcal{G}$ consists of $X \times X$-matrices whose rows and columns only have finitely many elements.
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Precise Definitions

**All discrete groups (Lück)**

\[ b_p^{(2)}(\Gamma) = \dim_{L(\Gamma)} H_p(\Gamma, L(\Gamma)) = \dim_{L(\Gamma)} \text{Tor}_p^{C\Gamma}(\mathbb{C}, L(\Gamma)) \in [0, \infty]. \]

**Measured groupoids (S.)**

\[ b_p^{(2)}(G) = \dim_{L(G)} \text{Tor}_p^{C\Gamma}(L^{\infty}(G^0), L(G)) \in [0, \infty]. \]

**Finite von Neumann algebras (Connes-Shlyakhtenko)**

\[ b_p^{(2)}(A) = \dim_{A \otimes A^{\text{op}}} \text{Tor}_p^{A \otimes A^{\text{op}}}(A, A \otimes A^{\text{op}}) \in [0, \infty]. \]
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The following theorem was first proved by Gaboriau for a different definition of $L^2$-Betti numbers (Cheeger-Gromov type rather than homological algebra).

**Theorem**

For any $\mu$-preserving $\Gamma$-action on a probability space $(X, \mu)$,

$$b_p^{(2)}(\Gamma) = b_p^{(2)}(X \rtimes \Gamma) \text{ for all } p \geq 0.$$ 

**Optimistic conjecture**

For every countable group one has

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Spectral sequence computations for $b_p^{(2)}(\Gamma)$

**Hochschild-Serre spectral sequence**

..computes $H_p(\Gamma; L(\Gamma))$ from $\Lambda$ and $Q$ for an extension

$$1 \to \Lambda \to \Gamma \to Q \to 1.$$  

$E_2^{p,q} = H_p(Q, H_q(\Lambda, L(\Gamma)))$

$E_1^{p,q} = P_p \otimes_{\mathbb{C}Q} H_q(\Lambda, L(\Gamma))$ where $\mathbb{C} \leftarrow P_*$ projective $\mathbb{C}Q$-resolution.

**Discouraging remark about "compute"**

It is extremely hard if the spectral sequence does not collapse!

**Prototype vanishing result**

Assume $b_p^{(2)}(\Lambda) = 0$ for $p > m$. Then $b_k^{(2)}(\Gamma) = 0$ for $k > m + \text{cd}_{\mathbb{C}}(Q)$. 
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Consider \( \Gamma \bowtie (X, \mu) \). Projective \( \mathbb{C}(X \rtimes \Gamma) \)-resolutions of \( L^\infty(X) \) can be used to compute \( b_p^{(2)}(X \rtimes \Gamma) = b_p^{(2)}(\Gamma) \).

Unfortunately, \( \mathbb{C}(X \rtimes \Gamma) \otimes_{\mathbb{C}\Gamma} - \) is not exact.

The following functor is exact and preserves projectives:

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The trivial module \( \mathbb{C} \) is mapped to \( L^\infty(X) \).

The projective dimension of \( L^\infty(X) \) is often smaller than \( \text{cd}_{\mathbb{C}}(\Gamma) \) in the quotient category.
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The **projective dimension** of $L^\infty(X)$ is often smaller than $\text{cd}_{\mathbb{C}}(\Gamma)$ in the quotient category.
For any infinite amenable group and $\Gamma \curvearrowright (X, \mu)$, $L^\infty(X)$ has a length 1 projective resolution in the quotient category (Connes-Feldman-Weiss)

The same holds for finite products of infinite amenable groups (Gaboriau).

More generally, Lattices in the same locally compact group have Morita equivalent quotient categories for suitable actions.

Let $\Gamma$ be a uniform lattice in semi-simple $G$ with finite center and no compact factors. For suitable $\Gamma \curvearrowright (X, \mu)$, the projective dimension of $L^\infty(X)$ in the quotient category is

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Short exact sequence

We define (slightly generalizing Feldman-Sutherland-Zimmer) the notion of a **short exact sequence** for discrete measured groupoids

\[ 1 \to G_1 \to G_2 \to G_3 \to 1. \]

It’s probably what you think it is plus ergodicity of \( G_1 \) with respect to almost every disintegration measure.

Informal theorem

- There is some graded \( \mathcal{U}(G_2) \)-module whose dimension equals \( b_\ast^{(2)}(G_2) \).

- There is a spectral sequence in terms of data of \( G_1 \) and \( G_2 \) that converges to this graded module.
Spectral sequence for discrete measured groupoids

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The following theorem was proved under the additional assumptions on $\Gamma/\Lambda$ (Lück) or the degree $d = 1$ (Gaboriau) before.

**Theorem**

Let $\Lambda \subset \Gamma$ be a normal subgroup of infinite index. If $b_p^{(2)}(\Lambda) = 0$ for $0 \leq p \leq d - 1$ and $b_d^{(2)}(\Lambda) < \infty$ then $b_p^{(2)}(\Gamma) = 0$ for $0 \leq p \leq d$.

**Theorem**

Consider $1 \rightarrow \Lambda \rightarrow \Gamma \rightarrow Q_0 \rightarrow 1$. Let

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Let $Q_0$ be measure equivalent to $Q_1$ (for example, $Q_0$, $Q_1$ lattices in the same locally compact group). Then

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Let $M$ be a closed aspherical $2n$-dimensional manifold that satisfies the Hopf-Singer conjecture, that is $b_p^{(2)}(\tilde{M}) = 0$ unless $p = n$. Let $F$ be a finite set of integers $\geq 2$. If $N$ is the total space of a fiber bundle

$$M \rightarrow N \rightarrow \prod_{g \in F} \Sigma g$$

then $N$ satisfies the Hopf-Singer conjecture.

- Let $m = \# F$. Note that $\prod_{g \in F} \Sigma g$ and $\prod_{g \in F} SL(2, \mathbb{Z})$ are lattices in the same Lie group; The latter has $cd_{\mathbb{C}} = m$.

- $\Rightarrow b_p^{(2)}(\tilde{N}) = b_p^{(2)}(\Gamma) = 0$ for $p > m + n$.

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By applying the spectral sequence to

$$1 \rightarrow G^{iso} \rightarrow G \rightarrow G^{equiv} \rightarrow 1$$

we obtain:

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Consider $\Gamma \bowtie (X, \mu)$. If the $L^2$-Betti numbers of almost every stabilizer vanish then also the $L^2$-Betti numbers of $\Gamma$.

The following is related to earlier results of Feldman, Sutherland and Zimmer, who prove a similar result for lattices in semi-simple groups.

Theorem

Let $b_p^{(2)}(\Gamma) \neq 0$ for some $p \geq 0$. The orbit equivalence relation of any free $\Gamma \bowtie (X, \mu)$ has no infinite, normal amenable subrelation.
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In a first non-trivial step we show that $b_p^{(2)}(G)$ is the $\mathcal{U}(G)$-dimension of the derived functor of the left exact functor

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\left\{ \mathcal{C}G\text{-modules} \right\}/\left\{ M; \dim_{L\infty}(G^0) M = 0 \right\} \rightarrow \{ \text{abelian groups} \}
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$M \mapsto \text{hom}(L(G^0), M)$.

evaluated at $\mathcal{U}(G)$.

It is possible to write $F$ as a composition of two functors. The desired spectral sequence is a Grothendieck spectral sequence with respect to that composition.

The analysis is well hidden behind the algebra. But showing that $F$ is the composition of the right functors is the real work.

Question: What about $L^2$-Betti numbers of von Neumann algebras?
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