A new tool for computing L^2-Betti numbers of groups

Roman Sauer Andreas Thom

[1] University of Chicago

[2] University of Göttingen

Banff, July 2007

L^2-Betti numbers – originally

Definition for Riemannian manifolds (Atiyah)

Let $\widetilde{M} \to M$ be the universal covering, and let $\mathscr{F} \subset \widetilde{M}$ be a $\pi_1(M)$-fundamental domain. Then define

$$b_p^{(2)}(\widetilde{M} : \pi_1(M)) = \int_{\mathscr{F}} \mathrm{tr}_{\mathbb{C}}\, e^{-t\Delta_p}(x, x) dvol(x).$$

Simplicial definition (Dodziuk)

For a (finite) simplicial complex K with $\Gamma = \pi_1(K)$, define $b_p^{(2)}(\widetilde{K} : \Gamma)$ as the Murray-von Neumann dimension of the Hilbert Γ-module

$$\bar{H}^p(\widetilde{K}, l^2(\Gamma)).$$

For a group Γ we set $b_p^{(2)}(\Gamma) = b_p^{(2)}(E\Gamma : \Gamma)$.

L^2-Betti numbers – originally

Definition for Riemannian manifolds (Atiyah)

Let $\widetilde{M} \to M$ be the universal covering, and let $\mathscr{F} \subset \widetilde{M}$ be a $\pi_1(M)$-fundamental domain. Then define

$$b_p^{(2)}(\widetilde{M} : \pi_1(M)) = \int_{\mathscr{F}} \operatorname{tr}_{\mathbb{C}} e^{-t\Delta_p}(x,x)\,dvol(x).$$

Simplicial definition (Dodziuk)

For a (finite) simplicial complex K with $\Gamma = \pi_1(K)$, define $b_p^{(2)}(\widetilde{K} : \Gamma)$ as the Murray-von Neumann dimension of the Hilbert Γ-module

$$\bar{H}^p(\widetilde{K}, l^2(\Gamma)).$$

For a group Γ we set $b_p^{(2)}(\Gamma) = b_p^{(2)}(E\Gamma : \Gamma)$.

Widening the scope of L^2-Betti numbers

- Lück and Farber defined different algebraic definitions to extend the L^2-Betti numbers to arbitrary Γ-spaces and groups. Lück's machinery allows the use of standard spectral sequences to compute L^2-Betti numbers.
- Gaboriau defined $b_p^{(2)}(\mathcal{R})$ for a measured equivalence relation $\mathcal{R}$. Later on, generalization to discrete measured groupoids.
- Example: Orbit equivalence relation of $\Gamma \curvearrowright (X, \mu)$, where (X, μ) is a probability space. In this case, $b_p^{(2)}(\mathcal{R}) = b_p^{(2)}(\Gamma)$. Any infinite amenable group is orbit equivalent to $\mathbb{Z}$.
- Connes-Shlyakhtenko defined $b_p^{(2)}(\mathcal{A})$ for an abitrary finite von Neumann algebra $\mathcal{A}$.

Widening the scope of L^2-Betti numbers

- Lück and Farber defined different algebraic definitions to extend the L^2-Betti numbers to arbitrary Γ-spaces and groups. Lück's machinery allows the use of standard spectral sequences to compute L^2-Betti numbers.
- Gaboriau defined $b_p^{(2)}(\mathcal{R})$ for a measured equivalence relation $\mathcal{R}$. Later on, generalization to discrete measured groupoids.
- Example: Orbit equivalence relation of $\Gamma \curvearrowright (X, \mu)$, where (X, μ) is a probability space. In this case, $b_p^{(2)}(\mathcal{R}) = b_p^{(2)}(\Gamma)$. Any infinite amenable group is orbit equivalent to $\mathbb{Z}$.
- Connes-Shlyakhtenko defined $b_p^{(2)}(\mathcal{A})$ for an abitrary finite von Neumann algebra $\mathcal{A}$.

Widening the scope of L^2-Betti numbers

- Lück and Farber defined different algebraic definitions to extend the L^2-Betti numbers to arbitrary Γ-spaces and groups. Lück's machinery allows the use of standard spectral sequences to compute L^2-Betti numbers.
- Gaboriau defined $b_p^{(2)}(\mathcal{R})$ for a measured equivalence relation $\mathcal{R}$. Later on, generalization to discrete measured groupoids.
- Example: Orbit equivalence relation of $\Gamma \curvearrowright (X, \mu)$, where (X, μ) is a probability space. In this case, $b_p^{(2)}(\mathcal{R}) = b_p^{(2)}(\Gamma)$. Any infinite amenable group is orbit equivalent to $\mathbb{Z}$.
- Connes-Shlyakhtenko defined $b_p^{(2)}(\mathcal{A})$ for an abitrary finite von Neumann algebra $\mathcal{A}$.

von Neumann algebras and dimension theory

Finite von Neumann algebras

- Finite von Neumann algebras are weakly closed $*$-subalgebras of some $\mathcal{B}(H)$ with a finite trace tr which has the trace property $\mathrm{tr}(ab) = \mathrm{tr}(ba)$.
- $L^\infty(X, \mu)$ with $\mathrm{tr}(f) = \int_X f d\mu$
- group von Neumann algebra: $L(\Gamma) = \mathcal{B}(l^2\Gamma)^\Gamma$ (Γ-equivariant bounded operators) with trace $\mathrm{tr}(a) = \langle a(1), 1\rangle$.

Dimension function for arbitrary modules (Lück)

There exists an additive dimension function

$$\dim_{\mathcal{A}} : \{\mathcal{A}\text{-modules}\} \to [0, \infty]$$

such that if $p \in \mathcal{A}$ is a projection then $\dim_{\mathcal{A}}(\mathcal{A}p) = \mathrm{tr}_{\mathcal{A}}(p)$.

von Neumann algebras and dimension theory

Finite von Neumann algebras

- Finite von Neumann algebras are weakly closed $*$-subalgebras of some $\mathcal{B}(H)$ with a finite trace tr which has the trace property $\mathrm{tr}(ab) = \mathrm{tr}(ba)$.
- $L^\infty(X, \mu)$ with $\mathrm{tr}(f) = \int_X f d\mu$
- group von Neumann algebra: $L(\Gamma) = \mathcal{B}(l^2\Gamma)^\Gamma$ (Γ-equivariant bounded operators) with trace $\mathrm{tr}(a) = \langle a(1), 1 \rangle$.

Dimension function for arbitrary modules (Lück)

There exists an additive dimension function

$$\dim_{\mathcal{A}} : \{\mathcal{A}\text{-modules}\} \to [0, \infty]$$

such that if $p \in \mathcal{A}$ is a projection then $\dim_{\mathcal{A}}(\mathcal{A}p) = \mathrm{tr}_{\mathcal{A}}(p)$.

Discrete measured groupoids

....don't care about groupoids...time to relax

Please don't! Discrete measured groupoids can be used effectively to compute L^2-Betti numbers of groups.

Discrete measured groupoids

Examples

- translation groupoid $X \rtimes \Gamma$ of a μ-preserving action $\Gamma \curvearrowright (X, \mu)$. If the action is free then $X \rtimes \Gamma$ is the orbit equivalence relation.
- holonomy groupoids of measured foliations (restricted to a transversal)

Groupoid ring and von Neumann algebra of a groupoid $\mathcal{G}$

- Groupoid ring $\mathbb{C}\mathcal{G}$ consists of finitely supported Borel functions $\mathcal{G} \to \mathbb{C}$ equipped with a convolution product. $L^\infty(\mathcal{G}^0)$ is $\mathbb{C}\mathcal{G}$-module.
- $\mathbb{C}\mathcal{G}$ carries a trace, and its von Neumann algebra completion is denoted by $L(\mathcal{G})$.
- If $\mathcal{G}$ is the orbit equivalence relation of $\Gamma \curvearrowright (X, \mu)$ then $\mathbb{C}\mathcal{G}$ consists of $X \times X$-matrices whose rows and colums only have finitely many elements.

Discrete measured groupoids

Examples

- translation groupoid $X \rtimes \Gamma$ of a μ-preserving action $\Gamma \curvearrowright (X, \mu)$. If the action is free then $X \rtimes \Gamma$ is the orbit equivalence relation.
- holonomy groupoids of measured foliations (restricted to a transversal)

Groupoid ring and von Neumann algebra of a groupoid $\mathcal{G}$

- Groupoid ring $\mathbb{C}\mathcal{G}$ consists of finitely supported Borel functions $\mathcal{G} \to \mathbb{C}$ equipped with a convolution product. $L^\infty(\mathcal{G}^0)$ is $\mathbb{C}\mathcal{G}$-module.
- $\mathbb{C}\mathcal{G}$ carries a trace, and its von Neumann algebra completion is denoted by $L(\mathcal{G})$.
- If $\mathcal{G}$ is the orbit equivalence relation of $\Gamma \curvearrowright (X, \mu)$ then $\mathbb{C}\mathcal{G}$ consists of $X \times X$-matrices whose rows and colums only have finitely many elements.

Precise Definitions

All discrete groups (Lück)

$$b_p^{(2)}(\Gamma) = \dim_{L(\Gamma)} H_p(\Gamma, L(\Gamma)) = \dim_{L(\Gamma)} \mathrm{Tor}_p^{\mathbb{C}\Gamma}\big(\mathbb{C}, L(\Gamma)\big) \in [0, \infty].$$

Measured groupoids (S.)

$$b_p^{(2)}(\mathcal{G}) = \dim_{L(\mathcal{G})} \mathrm{Tor}_p^{\mathbb{C}\mathcal{G}}\big(L^\infty(\mathcal{G}^0), L(\mathcal{G})\big) \in [0, \infty].$$

Finite von Neumann algebras (Connes-Shlyakhtenko)

$$b_p^{(2)}(\mathcal{A}) = \dim_{\mathcal{A} \overline{\otimes} \mathcal{A}^{op}} \mathrm{Tor}_p^{\mathcal{A} \otimes \mathcal{A}^{op}}\big(\mathcal{A}, \mathcal{A} \overline{\otimes} \mathcal{A}^{op}\big) \in [0, \infty]$$

Precise Definitions

All discrete groups (Lück)

$$b_p^{(2)}(\Gamma) = \dim_{L(\Gamma)} H_p(\Gamma, L(\Gamma)) = \dim_{L(\Gamma)} \mathrm{Tor}_p^{\mathbb{C}\Gamma}\left(\mathbb{C}, L(\Gamma)\right) \in [0, \infty].$$

Measured groupoids (S.)

$$b_p^{(2)}(\mathcal{G}) = \dim_{L(\mathcal{G})} \mathrm{Tor}_p^{\mathbb{C}\mathcal{G}}\left(L^\infty(\mathcal{G}^0), L(\mathcal{G})\right) \in [0, \infty].$$

Finite von Neumann algebras (Connes-Shlyakhtenko)

$$b_p^{(2)}(\mathcal{A}) = \dim_{\mathcal{A}\overline{\otimes}\mathcal{A}^{op}} \mathrm{Tor}_p^{\mathcal{A}\otimes\mathcal{A}^{op}}\left(\mathcal{A}, \mathcal{A}\overline{\otimes}\mathcal{A}^{op}\right) \in [0, \infty]$$

Precise Definitions

All discrete groups (Lück)

$$b_p^{(2)}(\Gamma) = \dim_{L(\Gamma)} H_p(\Gamma, L(\Gamma)) = \dim_{L(\Gamma)} \mathrm{Tor}_p^{\mathbb{C}\Gamma}\big(\mathbb{C}, L(\Gamma)\big) \in [0, \infty].$$

Measured groupoids (S.)

$$b_p^{(2)}(\mathcal{G}) = \dim_{L(\mathcal{G})} \mathrm{Tor}_p^{\mathbb{C}\mathcal{G}}\big(L^\infty(\mathcal{G}^0), L(\mathcal{G})\big) \in [0, \infty].$$

Finite von Neumann algebras (Connes-Shlyakhtenko)

$$b_p^{(2)}(\mathcal{A}) = \dim_{\mathcal{A}\,\overline{\otimes}\,\mathcal{A}^{op}} \mathrm{Tor}_p^{\mathcal{A}\otimes\mathcal{A}^{op}}\big(\mathcal{A}, \mathcal{A}\,\overline{\otimes}\,\mathcal{A}^{op}\big) \in [0, \infty]$$

Relations between three definitions

The following theorem was first proved by Gaboriau for a different definition of L^2-Betti numbers (Cheeger-Gromov type rather than homological algebra).

Theorem

For any μ-preserving Γ-action on a probability space (X, μ),

$$b_p^{(2)}(\Gamma) = b_p^{(2)}(X \rtimes \Gamma) \text{ for all } p \geq 0.$$

Optimistic conjecture

For every countable group one has

$$b_p^{(2)}(\Gamma) = b_p^{(2)}(L(\Gamma)) \text{ for all } p \geq 0.$$

Relations between three definitions

The following theorem was first proved by Gaboriau for a different definition of L^2-Betti numbers (Cheeger-Gromov type rather than homological algebra).

Theorem

For any μ-preserving Γ-action on a probability space (X, μ),

$$b_p^{(2)}(\Gamma) = b_p^{(2)}(X \rtimes \Gamma) \text{ for all } p \geq 0.$$

Optimistic conjecture

For every countable group one has

$$b_p^{(2)}(\Gamma) = b_p^{(2)}(L(\Gamma)) \text{ for all } p \geq 0.$$

Spectral sequence computations for $b_p^{(2)}(\Gamma)$

Hochschild-Serre spectral sequence

..computes $H_p(\Gamma; L(\Gamma))$ from Λ and Q for an extension

$$1 \to \Lambda \to \Gamma \to Q \to 1.$$

$E^2_{p,q} = H_p(Q, H_q(\Lambda, L(\Gamma))$
$E^1_{p,q} = P_p \otimes_{\mathbb{C}Q} H_q(\Lambda, L(\Gamma))$ where $\mathbb{C} \leftarrow P_*$ projective $\mathbb{C}Q$-resolution.

Discouraging remark about "compute"

It is extremely hard if the spectral sequence does not collapse!

Prototype vanishing result

Assume $b_p^{(2)}(\Lambda) = 0$ for $p > m$. Then $b_k^{(2)}(\Gamma) = 0$ for $k > m + \mathrm{cd}_{\mathbb{C}}(Q)$.

Spectral sequence computations for $b_p^{(2)}(\Gamma)$

Hochschild-Serre spectral sequence

..computes $H_p(\Gamma; L(\Gamma))$ from Λ and Q for an extension

$$1 \to \Lambda \to \Gamma \to Q \to 1.$$

$E^2_{p,q} = H_p(Q, H_q(\Lambda, L(\Gamma))$
$E^1_{p,q} = P_p \otimes_{\mathbb{C}Q} H_q(\Lambda, L(\Gamma))$ where $\mathbb{C} \leftarrow P_*$ projective $\mathbb{C}Q$-resolution.

Discouraging remark about "compute"

It is extremely hard if the spectral sequence does not collapse!

Prototype vanishing result

Assume $b_p^{(2)}(\Lambda) = 0$ for $p > m$. Then $b_k^{(2)}(\Gamma) = 0$ for $k > m + \mathrm{cd}_{\mathbb{C}}(Q)$.

Spectral sequence computations for $b_p^{(2)}(\Gamma)$

Hochschild-Serre spectral sequence

..computes $H_p(\Gamma; L(\Gamma))$ from Λ and Q for an extension

$$1 \to \Lambda \to \Gamma \to Q \to 1.$$

$E^2_{p,q} = H_p(Q, H_q(\Lambda, L(\Gamma))$
$E^1_{p,q} = P_p \otimes_{\mathbb{C}Q} H_q(\Lambda, L(\Gamma))$ where $\mathbb{C} \leftarrow P_*$ projective $\mathbb{C}Q$-resolution.

Discouraging remark about "compute"

It is extremely hard if the spectral sequence does not collapse!

Prototype vanishing result

Assume $b_p^{(2)}(\Lambda) = 0$ for $p > m$. Then $b_k^{(2)}(\Gamma) = 0$ for $k > m + \mathrm{cd}_{\mathbb{C}}(Q)$.

Small resolutions after passing to groupoids I

- Consider $\Gamma \curvearrowright (X,\mu)$. Projective $\mathbb{C}(X \rtimes \Gamma)$-resolutions of $L^\infty(X)$ can be used to compute $b_p^{(2)}(X \rtimes \Gamma) = b_p^{(2)}(\Gamma)$.
- Unfortunately, $\mathbb{C}(X \rtimes \Gamma) \otimes_{\mathbb{C}\Gamma} \_$ is not exact.
- The following functor is exact and preserves projectives:

$$\begin{tikzcd} \{\mathbb{C}\Gamma\text{-modules}\} \arrow[rr, "\otimes"] & & \{\mathbb{C}(X \rtimes \Gamma)\text{-modules}\} \arrow[d, "\mathrm{pr}"] \\ & & \{\mathbb{C}(X \rtimes \Gamma)\text{-modules}\}/\{M; \dim_{L^\infty(X)} M = 0\} \end{tikzcd}$$

The trivial module $\mathbb{C}$ is mapped to $L^\infty(X)$.

- The projective dimension of $L^\infty(X)$ is often smaller than $\mathrm{cd}_{\mathbb{C}}(\Gamma)$ in the quotient category.

Small resolutions after passing to groupoids I

- Consider $\Gamma \curvearrowright (X, \mu)$. Projective $\mathbb{C}(X \rtimes \Gamma)$-resolutions of $L^\infty(X)$ can be used to compute $b_p^{(2)}(X \rtimes \Gamma) = b_p^{(2)}(\Gamma)$.
- Unfortunately, $\mathbb{C}(X \rtimes \Gamma) \otimes_{\mathbb{C}\Gamma} \_$ is not exact.
- The following functor is exact and preserves projectives:

$$\begin{tikzcd} \{\mathbb{C}\Gamma\text{-modules}\} \arrow[r, "\otimes"] & \{\mathbb{C}(X \rtimes \Gamma)\text{-modules}\} \arrow[d, "\mathrm{pr}"] \\ & \{\mathbb{C}(X \rtimes \Gamma)\text{-modules}\}/\{M;\dim_{L^\infty(X)} M = 0\} \end{tikzcd}$$

  The trivial module $\mathbb{C}$ is mapped to $L^\infty(X)$.
- The projective dimension of $L^\infty(X)$ is often smaller than $\mathrm{cd}_{\mathbb{C}}(\Gamma)$ in the quotient category.

Small resolutions after passing to groupoids I

- Consider $\Gamma \curvearrowright (X, \mu)$. Projective $\mathbb{C}(X \rtimes \Gamma)$-resolutions of $L^\infty(X)$ can be used to compute $b_p^{(2)}(X \rtimes \Gamma) = b_p^{(2)}(\Gamma)$.
- Unfortunately, $\mathbb{C}(X \rtimes \Gamma) \otimes_{\mathbb{C}\Gamma} \_$ is not exact.
- The following functor is exact and preserves projectives:

$$\begin{tikzcd}
\{\mathbb{C}\Gamma\text{-modules}\} \arrow[r, "\otimes"] & \{\mathbb{C}(X \rtimes \Gamma)\text{-modules}\} \arrow[d, "\mathrm{pr}"] \\
 & \{\mathbb{C}(X \rtimes \Gamma)\text{-modules}\}/\{M; \dim_{L^\infty(X)} M = 0\}
\end{tikzcd}$$

  The trivial module $\mathbb{C}$ is mapped to $L^\infty(X)$.
- The projective dimension of $L^\infty(X)$ is often smaller than $\mathrm{cd}_{\mathbb{C}}(\Gamma)$ in the quotient category.

Small resolutions after passing to groupoids II

- For any infinite amenable group and $\Gamma \curvearrowright (X, \mu)$, $L^\infty(X)$ has a length 1 projective resolution in the quotient category (Connes-Feldman-Weiss)
- The same holds for finite products of infinite amenable groups (Gaboriau).
- More generally, Lattices in the same locally compact group have Morita equivalent quotient categories for suitable actions.
- Let Γ be a uniform lattice in semi-simple G with finite center and no compact factors. For suitable $\Gamma \curvearrowright (X, \mu)$, the projective dimension of $L^\infty(X)$ in the quotient category is

$$\dim(G/K) - (\mathbb{R}\text{-rank of } G)$$

whereas $\mathrm{cd}_{\mathbb{C}}(\Gamma) = \dim(G/K)$.

Small resolutions after passing to groupoids II

- For any infinite amenable group and $\Gamma \curvearrowright (X, \mu)$, $L^\infty(X)$ has a length 1 projective resolution in the quotient category (Connes-Feldman-Weiss)
- The same holds for finite products of infinite amenable groups (Gaboriau).
- More generally, Lattices in the same locally compact group have Morita equivalent quotient categories for suitable actions.
- Let Γ be a uniform lattice in semi-simple G with finite center and no compact factors. For suitable $\Gamma \curvearrowright (X, \mu)$, the projective dimension of $L^\infty(X)$ in the quotient category is

$$\dim(G/K) - (\mathbb{R}\text{-rank of } G)$$

whereas $\mathrm{cd}_{\mathbb{C}}(\Gamma) = \dim(G/K)$.

Small resolutions after passing to groupoids II

- For any infinite amenable group and $\Gamma \curvearrowright (X, \mu)$, $L^\infty(X)$ has a length 1 projective resolution in the quotient category (Connes-Feldman-Weiss)
- The same holds for finite products of infinite amenable groups (Gaboriau).
- More generally, Lattices in the same locally compact group have Morita equivalent quotient categories for suitable actions.
- Let Γ be a uniform lattice in semi-simple G with finite center and no compact factors. For suitable $\Gamma \curvearrowright (X, \mu)$, the projective dimension of $L^\infty(X)$ in the quotient category is

$$\dim(G/K) - (\mathbb{R}\text{-rank of } G)$$

whereas $\mathrm{cd}_{\mathbb{C}}(\Gamma) = \dim(G/K)$.

Spectral sequence for discrete measured groupoids

Short exact sequence

We define (slightly generalizing Feldman-Sutherland-Zimmer) the notion of a short exact sequence for discrete measured groupoids

$$1 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 1.$$

It's probably what you think it is plus ergodicity of $\mathcal{G}_1$ with respect to almost every disintegration measure.

Informal theorem

- There is some graded $\mathcal{U}(\mathcal{G}_2)$-module whose dimension equals $b_*^{(2)}(\mathcal{G}_2)$.
- There is a spectral sequence in terms of data of $\mathcal{G}_1$ and $\mathcal{G}_2$ that converges to this graded module.

Spectral sequence for discrete measured groupoids

Short exact sequence

We define (slightly generalizing Feldman-Sutherland-Zimmer) the notion of a short exact sequence for discrete measured groupoids

$$1 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 1.$$

It's probably what you think it is plus ergodicity of $\mathcal{G}_1$ with respect to almost every disintegration measure.

Informal theorem

- There is some graded $\mathcal{U}(\mathcal{G}_2)$-module whose dimension equals $b_*^{(2)}(\mathcal{G}_2)$.
- There is a spectral sequence in terms of data of $\mathcal{G}_1$ and $\mathcal{G}_2$ that converges to this graded module.

Applications I

The following theorem was proved under the additional assumptions on Γ/Λ (Lück) or the degree $d = 1$ (Gaboriau) before.

Theorem

Let $\Lambda \subset \Gamma$ be a normal subgroup of infinite index. If $b_p^{(2)}(\Lambda) = 0$ for $0 \leq p \leq d-1$ and $b_d^{(2)}(\Lambda) < \infty$ then $b_p^{(2)}(\Gamma) = 0$ for $0 \leq p \leq d$.

Theorem

Consider $1 \to \Lambda \to \Gamma \to Q_0 \to 1$. Let

$$b_p^{(2)}(\Lambda) = 0 \text{ for } p > m.$$

Let Q_0 be measure equivalent to Q_1 (for example, Q_0, Q_1 lattices in the same locally compact group). Then

$$b_p^{(2)}(\Gamma) = 0 \text{ for } p > m + \mathrm{cd}_{\mathbb{C}}(Q_1).$$

Applications I

The following theorem was proved under the additional assumptions on Γ/Λ (Lück) or the degree $d = 1$ (Gaboriau) before.

Theorem

Let $\Lambda \subset \Gamma$ be a normal subgroup of infinite index. If $b_p^{(2)}(\Lambda) = 0$ for $0 \leq p \leq d-1$ and $b_d^{(2)}(\Lambda) < \infty$ then $b_p^{(2)}(\Gamma) = 0$ for $0 \leq p \leq d$.

Theorem

Consider $1 \to \Lambda \to \Gamma \to Q_0 \to 1$. Let

$$b_p^{(2)}(\Lambda) = 0 \text{ for } p > m.$$

Let Q_0 be measure equivalent to Q_1 (for example, Q_0, Q_1 lattices in the same locally compact group). Then

$$b_p^{(2)}(\Gamma) = 0 \text{ for } p > m + \mathrm{cd}_{\mathbb{C}}(Q_1).$$

Applications II

Theorem

Let M be a closed aspherical $2n$-dimensional manifold that satisfies the Hopf-Singer conjecture, that is $b_p^{(2)}(\tilde{M}) = 0$ unless $p = n$. Let F be a finite set of integers ≥ 2. If N is the total space of a fiber bundle

$$M \to N \to \prod_{g \in F} \Sigma_g$$

then N satisfies the Hopf-Singer conjecture.

- Let $m = \#F$. Note that $\prod_{g \in F} \Sigma_g$ and $\prod_{g \in F} SL(2, \mathbb{Z})$ are lattices in the same Lie group; The latter has $\mathrm{cd}_{\mathbb{C}} = m$.
- $\Rightarrow b_p^{(2)}(\tilde{N}) = b_p^{(2)}(\Gamma) = 0$ for $p > m + n$.
- By Poincare duality, $b_p^{(2)}(\tilde{N}) = 0$ unless $p = m + n$.

Applications II

Theorem

Let M be a closed aspherical $2n$-dimensional manifold that satisfies the Hopf-Singer conjecture, that is $b_p^{(2)}(\tilde{M}) = 0$ unless $p = n$. Let F be a finite set of integers ≥ 2. If N is the total space of a fiber bundle

$$M \to N \to \prod_{g \in F} \Sigma_g$$

then N satisfies the Hopf-Singer conjecture.

- Let $m = \#F$. Note that $\prod_{g \in F} \Sigma_g$ and $\prod_{g \in F} SL(2, \mathbb{Z})$ are lattices in the same Lie group; The latter has $\mathrm{cd}_{\mathbb{C}} = m$.
- $\Rightarrow b_p^{(2)}(\tilde{N}) = b_p^{(2)}(\Gamma) = 0$ for $p > m + n$.
- By Poincare duality, $b_p^{(2)}(\tilde{N}) = 0$ unless $p = m + n$.

Applications III

By applying the spectral sequence to

$$1 \to \mathcal{G}^{iso} \to \mathcal{G} \to \mathcal{G}^{equiv} \to 1$$

we obtain:

Theorem

Consider $\Gamma \curvearrowright (X, \mu)$. If the L^2-Betti numbers of almost every stabilizer vanish then also the L^2-Betti numbers of Γ.

The following is related to earlier results of Feldman, Sutherland and Zimmer, who prove a similar result for lattices in semi-simple groups.

Theorem

Let $b_p^{(2)}(\Gamma) \neq 0$ for some $p \geq 0$. The orbit equivalence relation of any free $\Gamma \curvearrowright (X, \mu)$ has no infinite, normal amenable subrelation.

Applications III

By applying the spectral sequence to

$$1 \to \mathcal{G}^{iso} \to \mathcal{G} \to \mathcal{G}^{equiv} \to 1$$

we obtain:

Theorem

Consider $\Gamma \curvearrowright (X, \mu)$. If the L^2-Betti numbers of almost every stabilizer vanish then also the L^2-Betti numbers of Γ.

The following is related to earlier results of Feldman, Sutherland and Zimmer, who prove a similar result for lattices in semi-simple groups.

Theorem

Let $b_p^{(2)}(\Gamma) \neq 0$ for some $p \geq 0$. The orbit equivalence relation of any free $\Gamma \curvearrowright (X, \mu)$ has no infinite, normal amenable subrelation.

Final remarks about construction

- In a first non-trivial step we show that $b_p^{(2)}(\mathcal{G})$ is the $\mathcal{U}(\mathcal{G})$-dimension of the derived functor of the left exact functor

$$\{\mathbb{C}\mathcal{G}\text{-modules}\}/\{M; \dim_{L^\infty(\mathcal{G}^0)} M = 0\} \to \{\text{abelian groups}\}$$
$$M \mapsto \hom(L(\mathcal{G}^0), M).$$

  evaluated at $\mathcal{U}(\mathcal{G})$.

- It is possible to write F as a composition of two functors. The desired spectral sequence is a Grothendieck spectral sequence with respect to that composition.
- The analysis is well hidden behind the algebra. But showing that F is the composition of the right functors is the real work.
- Question: What about L^2-Betti numbers of von Neumann algebras?

Final remarks about construction

- In a first non-trivial step we show that $b_p^{(2)}(\mathcal{G})$ is the $\mathcal{U}(\mathcal{G})$-dimension of the derived functor of the left exact functor

$$\{\mathbb{C}\mathcal{G}\text{-modules}\}/\{M; \dim_{L^\infty(\mathcal{G}^0)} M = 0\} \to \{\text{abelian groups}\}$$
$$M \mapsto \hom(L(\mathcal{G}^0), M).$$

  evaluated at $\mathcal{U}(\mathcal{G})$.
- It is possible to write F as a composition of two functors. The desired spectral sequence is a Grothendieck spectral sequence with respect to that composition.
- The analysis is well hidden behind the algebra. But showing that F is the composition of the right functors is the real work.
- Question: What about L^2-Betti numbers of von Neumann algebras?

Final remarks about construction

- In a first non-trivial step we show that $b_p^{(2)}(\mathcal{G})$ is the $\mathcal{U}(\mathcal{G})$-dimension of the derived functor of the left exact functor

$$\{\mathbb{C}\mathcal{G}\text{-modules}\}/\{M; \dim_{L^\infty(\mathcal{G}^0)} M = 0\} \to \{\text{abelian groups}\}$$
$$M \mapsto \hom(L(\mathcal{G}^0), M).$$

  evaluated at $\mathcal{U}(\mathcal{G})$.
- It is possible to write F as a composition of two functors. The desired spectral sequence is a Grothendieck spectral sequence with respect to that composition.
- The analysis is well hidden behind the algebra. But showing that F is the composition of the right functors is the real work.
- Question: What about L^2-Betti numbers of von Neumann algebras?

Final remarks about construction

- In a first non-trivial step we show that $b_p^{(2)}(\mathcal{G})$ is the $\mathcal{U}(\mathcal{G})$-dimension of the derived functor of the left exact functor

$$\{\mathbb{C}\mathcal{G}\text{-modules}\}/\{M; \dim_{L^\infty(\mathcal{G}^0)} M = 0\} \to \{\text{abelian groups}\}$$
$$M \mapsto \hom(L(\mathcal{G}^0), M).$$

  evaluated at $\mathcal{U}(\mathcal{G})$.
- It is possible to write F as a composition of two functors. The desired spectral sequence is a Grothendieck spectral sequence with respect to that composition.
- The analysis is well hidden behind the algebra. But showing that F is the composition of the right functors is the real work.
- Question: What about L^2-Betti numbers of von Neumann algebras?