

Volume and L^2 -Betti numbers of aspherical manifolds

Roman Sauer

University of Chicago

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Definition of L^2 -Betti numbers

L^2 -Betti numbers – analytically [Atiyah]

Let $\tilde{M} \rightarrow M$ be the universal cover of a compact Riemannian manifold, and let $\mathcal{F} \subset \tilde{M}$ be a $\pi_1(M)$ -fundamental domain. Then define

$$b_i^{(2)}(M) = \lim_{t \rightarrow \infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}} e^{-t\Delta_i}(x, x) d\operatorname{vol}(x).$$

L^2 -Betti numbers – homologically [Farber; Lück]

Let M be an arbitrary space and $\Gamma = \pi_1(M)$. Let

$$b_i^{(2)}(M) = \dim_{L(\Gamma)} H_i(L(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M})) \in [0, \infty].$$

Dimension function for finite von Neumann algebras [Lück]

$\dim_{\mathcal{A}}(M)$ is defined for any finite von Neumann algebra \mathcal{A} with trace $\operatorname{tr} : \mathcal{A} \rightarrow \mathbb{C}$ and any \mathcal{A} -module. For example, $\dim_{\mathcal{A}}(\mathcal{A}p) = \operatorname{tr}(p)$.

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Some properties of L^2 -Betti numbers

Basic properties

- $\pi_1(M)$ finite $\Rightarrow b_i^{(2)}(M) = b_i(\tilde{M})/|\pi_1(M)|$
- $\sum_{i \geq 0} (-1)^i b_i^{(2)}(M) = \chi(M) = \sum_{i \geq 0} (-1)^i b_i(M)$.
- $\bar{M} \rightarrow M$ d -sheeted cover $\Rightarrow b_i^{(2)}(\bar{M}) = d \cdot b_i^{(2)}(M)$.

Vanishing and non-vanishing

- Amenable groups are a large class of groups including solvable ones. If M aspherical, i.e. \tilde{M} contractible, and $\pi_1(M)$ amenable, then $b_i^{(2)}(M) = 0$ [Cheeger-Gromov].
- If M is a $2n$ -dimensional hyperbolic manifold then $b_i^{(2)}(M) = 0$ unless $i = n$.

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Volume theorems

- Let M be an n -dimensional, closed, aspherical manifold.

Theorem 1 [Gromov; S.]

If (M, g) has a lower Ricci curvature bound $\text{Ricci}(M, g) \geq -(n-1)g$, then

$$b_i^{(2)}(M) \leq \text{const}_n \text{vol}(M, g) \quad \text{for every } i \geq 0.$$

- Gromov states this inequality in his book *Metric structures on Riemannian and non-Riemannian spaces* along with an idea (*randomization*) which he attributes to Connes.
- We provide the first complete proof of that inequality. The rigorous implementation of Gromov's idea uses tools and ideas from Gaboriau's theory of L^2 -Betti numbers of measured equivalence relations and spaces with groupoid actions of such.

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Corollary (Minimal volume estimate)

$$b_i^{(2)}(M) \leq \text{const}_n \text{minvol}(M)$$

Definition (Minimal volume)

$$\text{minvol}(M) = \inf\{\text{vol}(M, g); -1 \leq \sec(g) \leq 1\}.$$

Vanishing theorems

- Let M be an n -dimensional, closed, aspherical manifold.

Theorem 2 [S.]

If M is covered by open, amenable sets such that every point belongs to at most n sets, then

$$b_i^{(2)}(M) = 0 \quad \text{for every } i \geq 0.$$

- $U \subset M$ is called amenable if $\pi_1(U)$ maps to an amenable subgroup of $\pi_1(M)$.
- There is also a version of this theorem for arbitrary spaces.
- The proof uses a geometric construction, which is based on the Rokhlin theorem from ergodic theory, and the randomization idea mentioned before.

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Corollary (Gap theorem for vanishing of $b_i^{(2)}$)

There is a constant $\epsilon_n > 0$ only depending on n such that

$$\text{minvol}(M) < \epsilon_n \Rightarrow b_i^{(2)}(M) = 0 \quad \text{for every } i \geq 0.$$

- The implication “Theorem \Rightarrow Corollary” follows from Gromov’s work.

Proof of Theorem 2 – a first try

General technique of bounding the i -th L^2 -Betti number by equivariant coverings

- Let $\Gamma = \pi_1(M)$. Construct, using the geometric assumption somehow, a Γ -equivariant open covering \mathcal{U} of \tilde{M} .
- Let us say that \mathcal{U} is indexed by an index set $I \subset \Gamma \times \mathbb{N}$.
- One obtains $f \in \text{map}(\tilde{M}, \Delta(\Gamma \times \mathbb{N}))^\Gamma$ (nerve map).
- Let \mathcal{F}_i be a set of Γ -representatives of $\Delta(\Gamma \times \mathbb{N})^{(i)}$.
Let $C_i(f) \in \mathbb{N}$ be the number of i -cells in \mathcal{F}_i hit by $f(\tilde{M})$.
- Since $\tilde{M} \simeq E\Gamma$, there is an equivariant homotopy retract

$$\tilde{M} \xrightarrow{f} \Delta(\Gamma \times \mathbb{N})$$

- By rank considerations,

$$b_i^{(2)}(M) \leq C_i(f).$$

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- Let $C_i(f) \in \mathbb{N}$ be the number of equivariant i -cells hit by $f(\tilde{M})$. In other words, ...

• Since $\tilde{M} \simeq E\Gamma$, there is an equivariant homotopy retract

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Proof of Theorem 2 – Randomization

Connes-Gromov's philosophy – in probabilistic terms

Rephrasing what we tried before

- View $\Omega = \text{map}(\tilde{M}, \Delta(\Gamma \times \mathbb{N}))$ as a Borel space with a Γ -action.
- We tried to find a Γ -invariant **point measure** $f \in \Omega$ such that

$$b_i^{(2)}(M) \leq C_i(f) = \#(i\text{-cells in } \mathcal{F}_i \text{ hit by } f) \quad \text{is arbitrarily small.}$$

Randomized Problem

- View $C_i : \Omega \rightarrow \mathbb{Z}$ as a **random variable**.
- Instead of seeking a point measure, find any Γ -invariant probability measure μ on Ω such that the **expected value**

$$\mathbb{E}_{(\Omega, \mu)}(C_i) = \int_{\Omega} C_i(f) d\mu(f) \quad \text{is arbitrarily small.}$$

- Show that $b_i^{(2)}(M) \leq \mathbb{E}_{(\Omega, \mu)}(C_i)$.

Proof of Theorem 2 – Randomization

Connes-Gromov's philosophy – in probabilistic terms

Deterministic Problem

- View $\Omega = \text{map}(\tilde{M}, \Delta(\Gamma \times \mathbb{N}))$ as a Borel space with a Γ -action.
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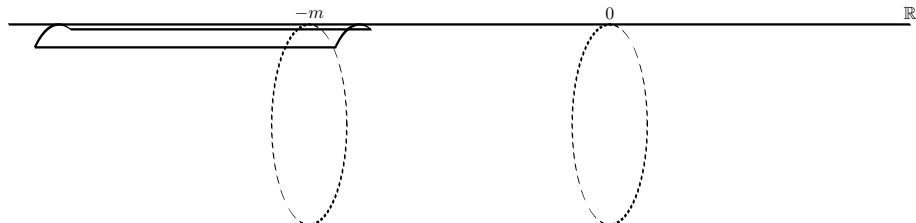
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Proof of Theorem 2 – the toy case $M = S^1$

Making the expected value $\mathbb{E}_{(\Omega, \mu)}(C_i)$ arbitrarily small for suitable μ

- Let $\mathbb{Z} = \langle t \rangle$ act on $X = S^1$ by an irrational angle α .
- Consider the diagonal action $\mathbb{Z} \curvearrowright X \times \tilde{M} = S^1 \times \mathbb{R}$.
(thanks to C. Löh for a lot of help with the picture!)

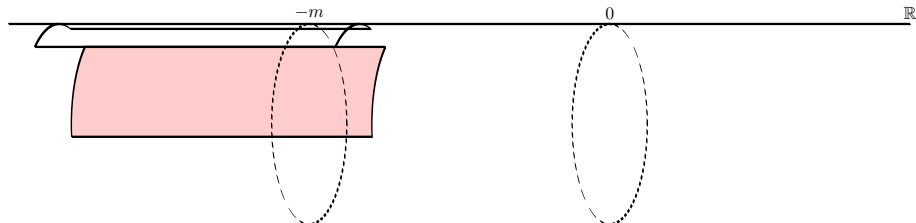


- Consider the “tile” on top of the cylinder with arclength α and length m .
- We now look at its \mathbb{Z} -orbit....

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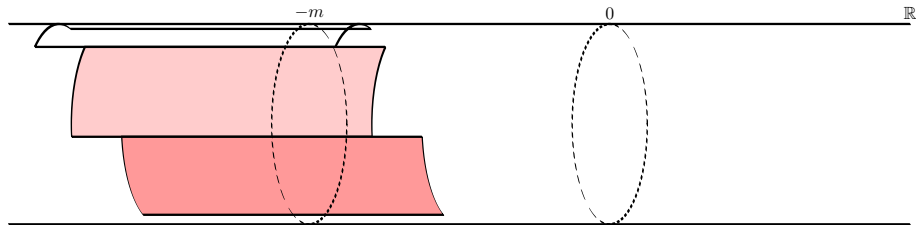


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- Here’s the t -translate.

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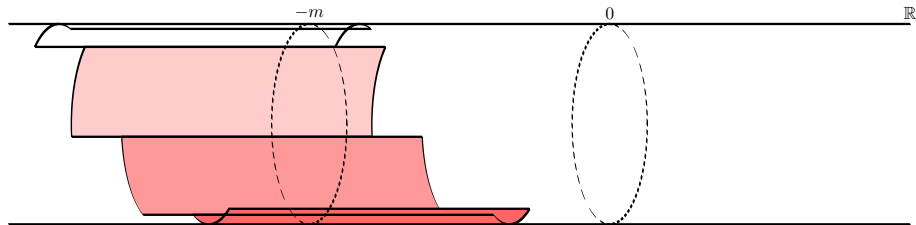


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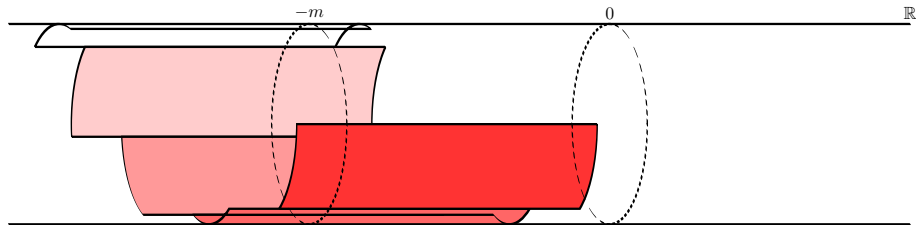


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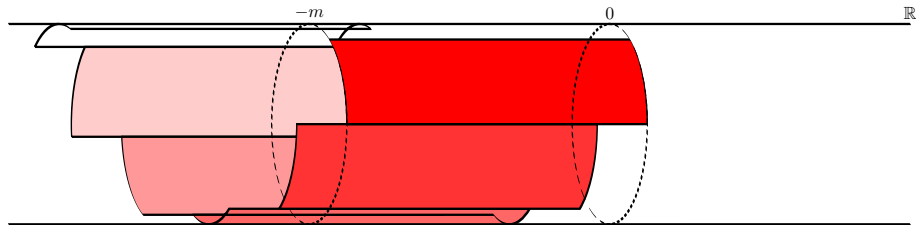


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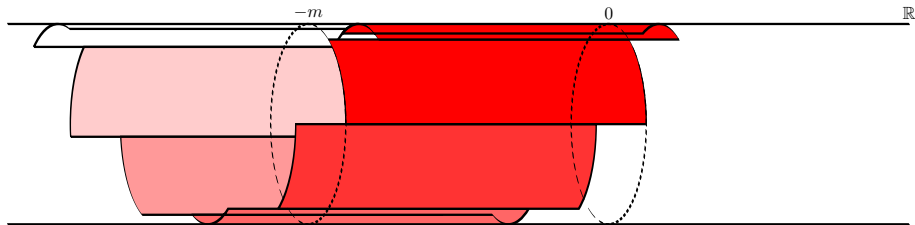


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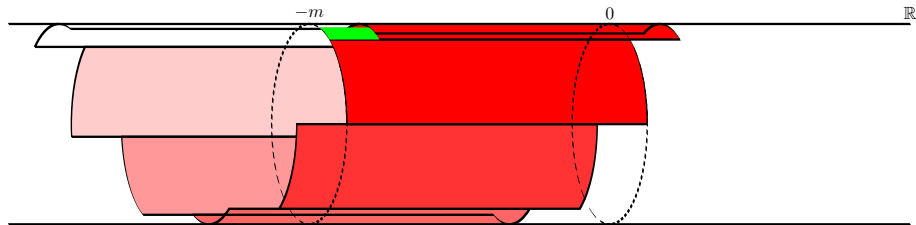


- After m steps the tiles almost close up, and the whole \mathbb{Z} -orbit almost tessellates $X \times \tilde{M}$.

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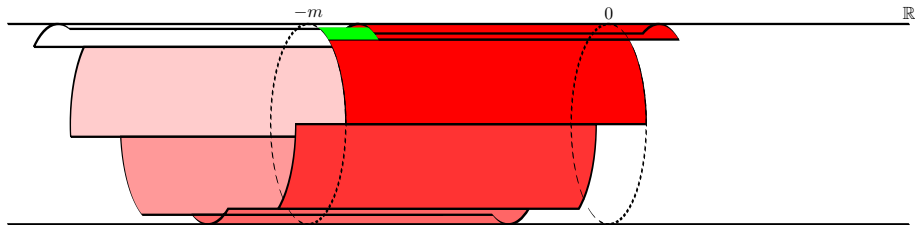


- Add the \mathbb{Z} -orbit of the small green tile, and one gets a tessellation of $X \times \tilde{M}$ indexed by $\mathbb{Z} \times \{1, 2\}$.

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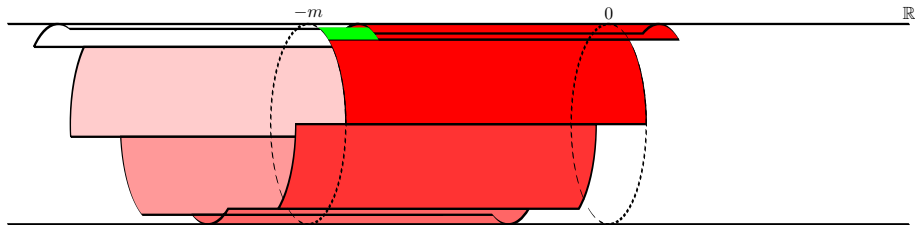


- Increase the size of the tiles in the induced tessellation of $\{z\} \times \tilde{M}$ a little to get a covering.
- For every $z \in S^1$ we obtain a map $\{z\} \times \tilde{M} \rightarrow \Delta(\mathbb{Z} \times \{1, 2\})$.

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- Equivalently, we have a map

$$\phi : S^1 \rightarrow \Omega = \text{map}(\tilde{M}, \Delta(\mathbb{Z} \times \{1, 2\})).$$

- The push-forward of the Haar measure on S^1 by ϕ is the **desired measure** μ .
- One has $\mathbb{E}_{(\Omega, \mu)}(C_i) \leq 2/m$.

L^2 -Betti numbers and actions on probability spaces

What's behind $b_i^{(2)}(M) \leq \mathbb{E}_{(\Omega, \mu)}(C_i)$? ▸

Group measure space construction [Murray-von Neumann]

The *group measure space construction*

$$L^\infty(\Omega, \mu) \rtimes \Gamma \hookrightarrow L^\infty(\Omega, \mu) \overline{\rtimes} \Gamma \hookrightarrow L(\Gamma)$$

is a completion of the algebraic crossed product $L^\infty(X) \rtimes \Gamma$ with respect to the trace

$$\text{tr}(\sum f_\gamma \gamma) = \int_\Omega f_1(x) d\mu(x).$$

L^2 -Betti numbers and Induction

$$\begin{aligned} b_i^{(2)}(M) &= \dim_{L^\infty(\Omega, \mu) \overline{\rtimes} \Gamma} H_i(L^\infty(\Omega, \mu) \overline{\rtimes} \Gamma \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M})) \\ &[= \dim_{L(\mathcal{R})} \text{Tor}_i^{\mathcal{C}\mathcal{R}}(L(\mathcal{R}), L^\infty(\Omega))] \end{aligned}$$

Proof of Theorem 2 – the general case

Where are the technical difficulties?

- Let $X = \{0, 1\}^\Gamma$. For each amenable $U \subset M$ construct a certain tessellation on each $X \times p^{-1}(U)$ and combine them.
Crucial ingredient: the generalized *Rokhlin lemma* from ergodic theory [Ornstein-Weiss].
- \rightsquigarrow Γ -invariant measure μ on $\Omega = \text{map}(\tilde{M}, \Delta(\Gamma \times \mathbb{N}))$ such that $\mathbb{E}_{(\Omega, \mu)}(C_n)$ is arbitrarily small.
- \rightsquigarrow construct a fundamental class $\sum f_i \otimes \sigma_i$ in

$$H_n(L^\infty(\Omega) \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M})) \xleftarrow{\cong} H_n(\mathbb{R} \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M})) = H_n(M, \mathbb{R}).$$

such that $\sum \mu(\text{supp}(f_i))$ is arbitrarily small.

- $\rightsquigarrow b_i^{(2)}(M)$ arbitrarily small [Schmidt].

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