Randomization methods in the theory of L^2-Betti numbers

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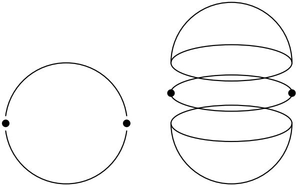
It starts with the Euler characteristic...

- The **Euler characteristic** of a (finite) cellular complex K is

$$\chi(K) = \#(\text{0-cells}) - \#(\text{1-cells}) + \#(\text{2-cells}) - \dots$$



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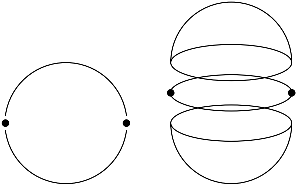

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- The n-th **Betti number** $b_n(M) \in \mathbb{N} \cup \{0\}$ of a topological space M is the rank of the n-th homology $H_n(M)$.
- **Euler-Poincaré formula:**

$$\chi(M) = \sum_{n \geq 0} (-1)^n b_n(M).$$

Towers of finite covers

- If $\bar{M} \to M$ is a d-**sheeted cover**, then $\chi(\bar{M}) = d \cdot \chi(M)$. One can lift the cell structure on M to $\bar{M}$ such that the pre-image of each cell in M consists of d cells in $\bar{M}$.
- Let $\pi = \pi_1(M)$ be the fundamental group of M. If $\Gamma \triangleleft \pi$, then there is a regular $[\pi : \Gamma]$-sheeted cover $\bar{M} = \Gamma \backslash \widetilde{M}$ of M.

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- Consider a **tower of finite index subgroups** $\pi = \Gamma_0 \triangleright \Gamma_1 \triangleright \Gamma_2 \triangleright \ldots$ with trivial intersection. The **sequence of finite covers** $M_i = \Gamma_i \backslash \widetilde{M}$ "approximates" the **universal cover** $\widetilde{M}$.
- Euler-Poincaré formula:

$$\chi(M) = \frac{1}{[\pi : \Gamma_i]} \cdot \chi(M_i) = \sum_{n \geq 0} (-1)^n \underbrace{\frac{b_n(M_i)}{[\pi : \Gamma_i]}}_{\text{Convergence as } i \to \infty \text{ ??}}$$

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Theorem (Lück)

The limit $\mathrm{b}_n^{(2)}(M) = \lim_{i \to \infty} \frac{1}{[\pi:\Gamma_i]} \cdot b_n(M_i)$ *is the n-th* L^2-**Betti number**.

History of L^2-Betti numbers

Definition (Atiyah 1976)

Let M be a closed Riemannian manifold and $F \subset \widetilde{M}$ a fundamental domain of $\pi_1(M) \curvearrowright \widetilde{M}$. Then set

$$\mathrm{b}_n^{(2)}(M) = \lim_{t \to \infty} \int_F \mathrm{trace}_{\mathbb{C}} \big(\underbrace{e^{-t\Delta_n}(x,x) \colon \mathrm{Alt}^n(T_x M) \circlearrowleft}_{\text{heat kernel on } n\text{-forms}} \big) \, \mathrm{dvol}(x).$$

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- Dodziuk defined $\mathrm{b}_n^{(2)}(K)$ for every finite simplicial complex K.
- For a group Γ you want to set

  $$\mathrm{b}_n^{(2)}(\Gamma) = \mathrm{b}_n^{(2)}(B\Gamma),$$

  where $B\Gamma$ is the **classifying space** of Γ. Cheeger-Gromov (1986) define $\mathrm{b}_n^{(2)}(\Gamma)$ even if $B\Gamma$ has no finite model.
- Lück (1999) gives the most general definition using his **dimension theory of modules over finite von Neumann algebras**.

More recent developments

- Gaboriau (2002) defined L^2-Betti numbers of spaces fibering over a probability space endowed with a (groupoid) action of a measured equivalence relation.
- Example of such a relation: the **orbit equivalence relation** of a **free measure preserving** action of a countable group on a probability space, e.g. $\Gamma \curvearrowright X = \{0,1\}^\Gamma$.

Theorem (Gaboriau)

Two countable groups Γ and Λ that have such actions on a probability space yielding the same orbit equivalence relation have the same L^2-Betti numbers $b_n^{(2)}(\Gamma) = b_n^{(2)}(\Lambda)$.

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- We can compute $\chi(M)$ and $b_n^{(2)}(M)$ bringing the dynamics of measure preserving $\pi_1(M)$-actions into play!
- Let's illustrate that with the Euler characteristic in a toy case!

Euler characteristic and m.p. actions

- Consider a manifold or a space M with fundamental group $\Gamma = \pi_1(M)$. Let $\Gamma \curvearrowright (X, \mu)$ be a free μ-preserving action.
- Consider an **equivariant measurable family of triangulations** T_x **of the universal cover** $\widetilde{M}$ parametrized over $x \in X$. All the i-simplices form a Γ**-invariant measurable subset**

$$T^{(i)} := \bigcup_{x \in X} \{x\} \times T_x^{(i)} \subset \underbrace{X \times \mathrm{map}(\Delta^i, \widetilde{M})}_{\Gamma \text{ acts diagonally}}.$$

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Choose measurable fundamental domains $F^{(i)}$ for $\Gamma \curvearrowright X \times \mathrm{map}(\Delta^i, \widetilde{M})$. Then:

$$\chi(M) = \sum_{i \geq 0} (-1)^i \int_X \left| T_x^{(i)} \cap F^{(i)} \right| d\mu(x).$$

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- If $T_x = T$ is a constant family of triangulations, this is trivial!
- Are there interesting non-constant families of triangulations?

A toy case

- Consider the case $M = S^1$, $\Gamma = \pi_1(M) = \mathbb{Z}$. Let the generator of Γ act on $(X, \mu) = (S^1, \mathrm{Haar})$ by a rotation of angle $\alpha \approx 2\pi/m$.
- We describe an equivariant partition of $X \times \widetilde{M} = S^1 \times \mathbb{R}$ by sets of the form (Borel subset $\times$ cell). On each $\{x\} \times \widetilde{M}$ you will then see a triangulation.

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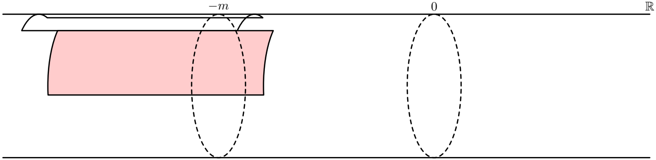
- Let's start! You see the set $[0, \alpha] \times [-2m + 1, -m + 1]$ and its first translate by the generator of $\Gamma = \mathbb{Z}$:



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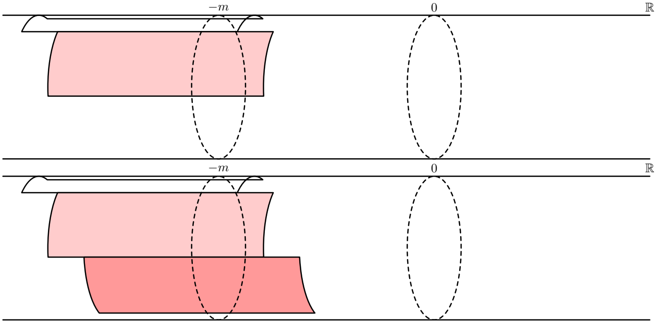
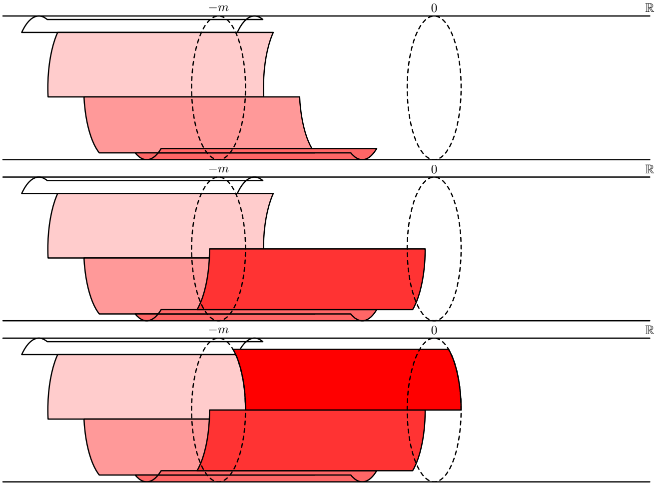


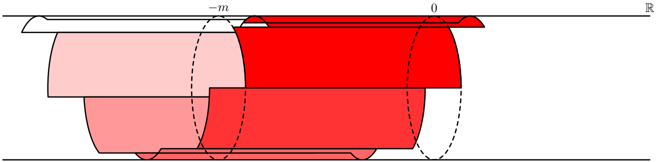
A toy case

- More translates...



A toy case

- We almost obtain a partition of the cylinder $S^1 \times \mathbb{R}$ but because of $m\alpha < 2\pi$ the translates do not quite close up after m steps.



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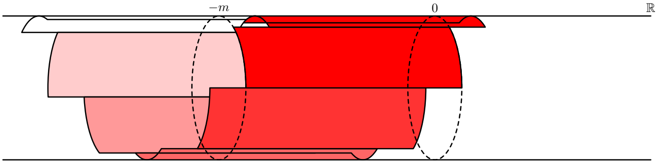
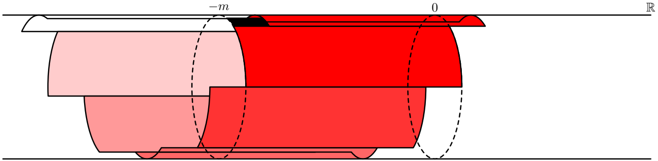


- We have to introduce another tile (black in the picture) whose Γ-orbit together with the orbit of the first tile partitions $S^1 \times \mathbb{R}$.



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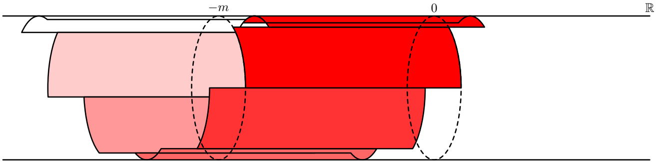

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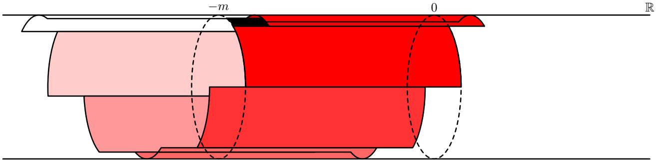
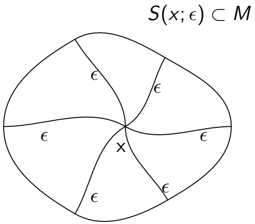


- $\rightsquigarrow$ Vanishing of $\chi(\Gamma)$ for amenable Γ (Rokhlin's theorem)

Reminder on curvature and asphericity

- For $\dim M = 2$: the **sectional curvature** of M is a function $K\colon M \to \mathbb{R}$ with
$$\mathrm{length}(S(x;\epsilon)) = 2\pi\epsilon - \frac{\pi}{3}K(x)\epsilon^3 + o(\epsilon^3).$$
For $\dim M \geq 2$ one has $K\colon \mathrm{Gr}(2, TM) \to \mathbb{R}$.
- The **Ricci curvature** $\mathrm{Ricci} : TM \to \mathbb{R}$ at $v \in T_xM$ is the average $\sum_{i=1}^{n-1} K(v, w_i)$, where $\{v, w_1, \ldots, w_n\}$ is an ON-basis of T_xM.
- A space is **aspherical** if its universal cover is contractible.
- Manifolds that admit a Riemannian metric of non-positive sectional curvature are aspherical.



Volume and L^2-Betti numbers

Theorem (Gromov, S)

Let M be an n-dimensional closed aspherical manifold. If g is a Riemannian metric with $\operatorname{Ricci}(M,g) \geq -(n-1)g$, *then*

$$\mathrm{b}_i^{(2)}(M) \leq \mathrm{const}_n \, \mathrm{vol}(M,g) \qquad \textit{for all } i \geq 0.$$

In particular, $\mathrm{b}_i^{(2)}(M) \leq \mathrm{const}_n \, \mathrm{minvol}(M)$, *where*

$$\mathrm{minvol}(M) = \inf\big\{\mathrm{vol}(M,g); \; -1 \leq K(g) \leq 1\big\}.$$

is the minimal volume of M.

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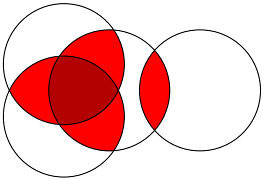
Theorem (S)

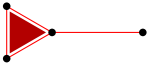
For every $n \in \mathbb{N}$ there is $\epsilon_n > 0$ such that every n-dimensional closed aspherical manifold M satisfies:

$$\mathrm{minvol}(M) < \epsilon_n \Longrightarrow b_i^{(2)}(M) = 0 \qquad \text{for all } i \geq 0.$$

A naive proof attempt

- **Bishop-Gromov inequality**: We obtain a cover $\mathcal{U}$ of $\widetilde{M}$ by 3-balls with multiplicity $<$ const$_n$ by taking 3-balls around points in a **maximal** 1-separated net.
- $\rightsquigarrow f : \widetilde{M} \to \mathit{nerve}(\mathcal{U})$ with Lipschitz constant $<$ const$_n$ and image in the n-skeleton of the nerve.
- If $\mathcal{U}$ is $\pi_1(M)$-equivariant, then $f(\widetilde{M})$ hits at most const$_n$ vol(M) orbits of n-cells.
- Asphericity yields that f is an equivariant homotopy retract, leading to a proof of the inequality.
- **Problem**: $\mathcal{U}$ cannot be made $\pi_1(M)$-equivariant!





Randomizing the previous strategy

- Consider a free μ-preserving action of $\Gamma = \pi_1(M)$ on a probability space (X, μ).
- Bishop-Gromov inequality again: **One can construct a Γ-equivariant cover $\mathcal{U} = \{A_i \times B_i\}_{i \in I}$ of $X \times \widetilde{M}$**, where $A_i \subset X$ measurable and $B_i \subset \widetilde{M}$ is a 3-ball, with multiplicity $< \mathrm{const}_n$.

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