Expander Graphs in Theoretical and Applied Contexts

Wintersemester 2022/23, Dr. Julia Heller and Dr. Martin Nitsche

Talk 1 – Graphs and expander graphs

Introduce basic definitions of graphs, show examples. Introduce expansion in graphs following [HLW06], explain the correspondence expander graphs vs. expander families, present and prove the Expander Mixing Lemma [HLW06, Lemma 2.5].

Remarks: Caution! The definition of expansion in [Gar12] differs from the one in [HLW06]. See [Gar12, Prop. 1.36] for the connection.
References: [HLW06, Sections 2.1, 2.3, Le. 2.5]; [Gar12, Sections 1.1, 1.2., 1.4]; [Kow16, Sections 2.1, 2.2]

Talk 2 – Existence of expander families and Margulis' construction

Give a sketch of the probabilistic proof of existence of expander families, following [Gar12]. Show examples of expander families. Explain Margulis' construcion of an expander family and give the ideas of the proof.

Remarks: The mentioned section 4.5 in [HLW06] will be discussed in talk 6. Use the results from section 4.5 without proof.
References: [Gar12, Section 1.5, and 4.3. for examples]; [HLW06, Sections 2.2, 8]; [Kim21, Talk 5]

Talk 3 – Random walks

Introduce the concept of random walks on graphs. Give examples. Introduce the adjacency matrix and discuss how its eigenvalues relate to the properties of the graph and the random walk on it. Give proofs wherever reasonable. State [HLW06, Theorem 4.11] without proof.

Remarks: The proof of [HLW06, Theorem 4.11] is part of talk 6.
References: [Gar12, Sections 2.1, 2.2]; [HLW06, Section 3.1]

Talk 4 – Application to probabilistic algorithms

Introduce the concept of probabilistic algorithms. Present an example of such an algorithm. Motivate why the random bits needed for probabilistic algorithms can be viewed as a scarce ressource. Explain how expander graphs can be used to reduce the required amount of random bits. Prove that this works [Gar12, Lemmas 2.17, 2.18 and Theorem 2.16].

References: [Gar12, Section 2.3]; [HLW06, Sections 3.2, 3.3]

Talk 5 – Application to error correcting codes

Introduce the concept of error correcting codes. Explain how such codes can be obtained as linear codes from bipartite graphs. Define the notion of a bipartite expander graph, but do not give an explicit construction. Prove that bipartite expander graphs give rise to efficient error correcting codes [HLW06, Theorems 12.8 and 12.9].

References: [HLW06, Sections 12.1, 12.2, 12.5]; [Sve14, Section 2]

Talk 6 – Expansion and the spectral gap

Explain the correspondence of the expansion ratio of a graph with the Cheeger constant of to Riemannian manifold. Explain and prove [HLW06, Theorem 4.11] and discuss that the bounds are tight.

Remarks: Do not present details or proofs from Riemannian geometry, but rather explain the intuition of the relation of the concepts. Talk 2 uses results from [HLW06, Section 4.5].
References: [HLW06, Sections 2.5, 4.3.1, 4.4, 4.5]; [Gar12, Section 2.2, Rem. 1.35]; [Kim21, Talk 2]

Talk 7 – Alon-Boppana lower bound and Ramanujan graphs*

Explain the Alon-Boppana lower bound and sketch a proof (choose one of [HLW06, 5.2.2 or 5.2.3]). Introduce Ramanujan graphs and discuss why they are best possible in the light of Alon-Boppana. Discuss [Kow16, Theorem 4.2.3]. As appropriate, introduce 2-lifts and explain the example for [Kow16, Theorem 4.2.3].

Remarks: The challenge of this talk is to get an overview of the broad topic. You may choose (in consultation with us) on which part of the suggested material you wish to focus on in your talk.
References: [HLW06, Sections 2.5, intro to 5, 5.2.1–5.2.3, 5.3, as appropriate: 6]; [Kow16, Section 4.2]; [Lub17]; [Kim21, Talk 4]

Talk 8 – Zig-zag products

Introduce the zig-zag and replacement products for graphs and prove the Zig-Zag-Theorem. Explain the construction of a family of expander graphs using the zig-zag product.

References: [HLW06, Section 9 up to and including 9.3], [Olv10], [Kim21, Talk 6]

Talk 9 – Cayley expander graphs*

Introduce Cayley graphs and the semi-direct product, give examples of both. Show the connection of the zig-zag product of graphs and the semi-direct product of groups. Present (one or two) constructions of Cayley expander graph families with the introduced techniques.

Remarks: The challenge of this talk is to get an overview of the broad topic. You may choose (in consultation with us) on which part of the suggested material you wish to focus on in your talk.
References: [HLW06, Chapter 11, in particular Sections 11.2, 11.3, as appropriate: 11.4]; [Kow16, Sections 2.3, 3.5]; [Gar12, Section 1.6];

Talk 10 – Expander graphs from property (T)*

Give a short introduction to the concept of representations of groups as unitary operators on Hilbert spaces (only for discrete groups, not topological groups). Define property (T) via Kazhdan pairs. Explain with proof how finite quotients of property (T) groups give expander graphs. Use SL(3,Z) as an example without giving proofs.

References: [Kow16, Section 4.4, pp. 107–108]; see also [Gar12] and [BHV08]

Talk 11 – Proving property (T) with the computer*

State the alternative definition of property (T) via spectral gaps of the Laplacian Δ. Sketch why this is equivalent to the definition via Kazhdan pairs (use the spectral theorem for bounded selfadjoint operators as a black box). Prove that the existence of a sum of squares decomposition of $\Delta^2 - \varepsilon\Delta$ implies a spectral gap. Explain that such a decomposition can be found with the computer.

References: start with [NT15]; see [BHV08] for background on property (T)

Talk 12 – Application to metric embeddings*

Prove the lower bound of the Barzdin-Kolmogorov graph-embedding theorem. Explain the statement of the Linial-London-Rabinovich theorem. Give a proof sketch of this theorem. Try to avoid SDP duality theory as much as possible.

References: [Kow16, Section 5.1]; [HLW06, Sections 13.2, 13.3]; background on SDP duality can be found in [BTN01, Chapter 2]

Remarks

- Talks marked with * contain advanced material.
- Use examples to illustrate statements and proofs whenever possible. Recall that your audience is new to the topic – examples and revision of definitions help your classmates to follow and understand.
- [Kim21] are lecture notes based on [HLW06].
- [Gar12] containes many useful remarks and examples, [Kow16] containes many examples and a detailed introduction. It could be helpful to have a look at these references, even if they are not explicitly listed in your talk's literature.
- Our main source [HLW06] containes many references in the text, which we do not list seperately in the description of the talks. You may want to have a look at them nevertheless.

References

[BHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan's Property* (T), New mathematical monographs 11, Cambridge University Press, 2008.

[BTN01] Aharon Ben-Tal and Arkadi Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, MPS-SIAM Series on Optimization, Society for Industrial Mathematics, 2001.

[Gar12] Giles Gardam, *Expander Graphs and Kazhdan's Property (T)*, University of Sydney, 2012, `https://www.gilesgardam.com/papers/expanders_and_t.pdf`. Honours thesis.

[HLW06] Shlomo Hoory, Nathan Linial, and Avi Wigderson, *Expander graphs and their applications*, Bulletin of the American Mathematical Society **43** (2006), available at `https://www.cs.huji.ac.il/~nati/PAPERS/expander_survey.pdf`.

[Kim21] Jaehoon Kim, *Lecture notes on Expanders* (2021), available at `http://homepages.warwick.ac.uk/staff/H.Liu.9/sdu2021-summer.html`.

[Kow16] Emmanuel Kowalski, *Expander graphs* (2016), available at `https://people.math.ethz.ch/~kowalski/expander-graphs.pdf`.

[Löh17] Clara Löh, *Geometric Group Theory: An Introduction*, Universitext (2017), available at `https://link.springer.com/book/10.1007/978-3-319-72254-2`.

[Lub17] Alexander Lubotzky, *Ramanujan Graphs* (2017), available at `https://arxiv.org/abs/1711.06558`.

[NT15] Tim Netzer and Andreas Thom, *Kazhdan's property (T) via semidefinite optimization*, Exp. Math. **24** (2015), no. 3, 371–374, available at `https://arxiv.org/abs/1411.2488`.

[Olv10] Neil Olver, *The zig-zag product* (2010), available at `https://www.math.mcgill.ca/goren/667.2010/Neil.pdf`.

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson, *Entropy waves, the zig-zag graph product, and new constant-degree expanders*, Ann. of Math. (2) **155** (2002), no. 1, 157–187, available at `https://www.jstor.org/stable/3062153`.

[Sve14] Ola Svensson, *Topics in Theoretical Computer Science* (2014), available at `https://theory.epfl.ch/courses/topicstcs/Lecture3.pdf`. Lecture notes.

Further reading and watching

- *Expander Graphs - A very brief introduction*, Giles Gardam, video clip, 2013, `https://www.youtube.com/watch?v=aOyoVckhaGc`
- *Expander Summer school*, Jaehoon Kim, video clips and lecture notes, Shandong University, 2021 `http://homepages.warwick.ac.uk/staff/H.Liu.9/sdu2021-summer.html`
- *What is . . . an expander?*, Peter Sarnak, Notices Amer. Math. Soc., 51(7):762–763, 2004, `https://www.ams.org/notices/200407/what-is.pdf`
- *Margulis' construction of a family of expander graphs*, Survey on the occasion of the 2020 Abel Prize Award for Gregory Margulis, `https://abelprize.no/sites/default/files/2021-04/Margulis_construction_expander_english.pdf`
- *Graph innvariants*, Survey on the occasion of the 2021 Abel Prize Award for László Lovász and Avi **W**igderson (who is the one of the authors of [HLW06]), `https://abelprize.no/sites/default/files/2021-09/Graph_invariants.pdf`