THE ASYMPTOTIC GROWTH OF TWISTED TORSION

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ABSTRACT. We prove a general limit formula for twisted analytic torsion which implies both [1, Theorem 4.5] and [8, Theorem 1.1].

CONTENTS

1. ASYMPTOTIC OF TWISTED HEAT KERNELS

Let M be a closed smooth manifold of dimension n. We fix a Riemannian metric on M. Let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be an acyclic representation on a finite dimensional real vector space. Let $E_\rho = \pi_1(M)\backslash(\widetilde{M} \times E)$ be the associated flat bundle. We choose a metric h on E_ρ.[1] It induces a $\mathbb{R}$-linear isomorphism $\# : E_\rho \to E_\rho^*$ where E_ρ^* is the dual vector bundle. Let $\Lambda^p(E)$ be the space of smooth p-forms on M with values in E_ρ, i.e. the space of smooth sections of $\Lambda^p(T^*M) \otimes E_\rho$. Then $\#$ extends to an isomorphism

$$\# : \Lambda^p(E) \to \Lambda^p(E^*)$$

for each p. Furthemore, the Riemannian metric on M defines a linear mapping

$$* : \Lambda^p \to \Lambda^{n-p}(E)$$

for each p which satisfies $** = (-1)^{p(n-p)}$ on $\Lambda^p(E)$, and $*$ and $\#$ commute. The usual exterior product of differential forms combined with $\mathrm{tr} : E_\rho \otimes E_\rho^* \to \mathbb{R}$ induces an exterior product on for vector-valued forms:

$$\wedge : \Lambda^p(E) \otimes \Lambda^q(E^*) \to \Lambda^{p+q}(M).$$

Then an inner product on $\Lambda^p(E)$ is defined by

$$\langle \omega, \omega' \rangle = \int_M \omega \wedge * \circ \# \omega'.$$

We denote by $L^2\Lambda^p(E)$ the completion of $\Lambda^p(E)$ with respect to this inner product. Since E_ρ is flat, we have the de Rham complex

$$\Lambda^0(E) \xrightarrow{d_0} \Lambda^1(E) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n(E).$$

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[1] Note that we don't (in fact can't in general) ask h to be compatible with the flat connection.

The formal adjoint of d_p on $\Lambda^p(E)$ is given by

$$\delta_p = (-1)^{np+n+1} * \circ \#^{-1} d_p \# \circ *.$$

Now we define the Laplacian $\Delta_p = \Delta_p(\rho)$ on p-forms by

$$\Delta_p = \delta_p d_p + d_{p-1}\delta_{p-1}.$$

We denote by $\lambda_p(\rho)$ the lowest eigenvalue of $\Delta_p(\rho)$.

Note that we may as well consider the Laplacian $\Delta_p^{(2)} = \Delta_p^{(2)}(\rho)$ on p-forms with values in E. We write $e^{-t\Delta_p^{(2)}}(x,y)$ for the corresponding heat kernel; it is a C^∞-function for $(x,y,t) \in \widetilde{M} \times \widetilde{M} \times \mathbb{R}_+^*$ with values in $\mathrm{End}(\Lambda_y^p(E), \Lambda_x^p(E)$ so that:

$$(e^{-t\Delta_p^{(2)}} f)(x) = \int_{\widetilde{M}} e^{-t\Delta_p^{(2)}}(x,y) f(y) dy,$$

for any square-integrable p-form f on $\widetilde{M}$ with values in E and any positive t. We denote by $\lambda_p^{(2)}(\rho)$ the bottom of the spectrum of $\Delta_p^{(2)}(\rho)$.

1.1. From now on we assume that both $\lambda_p(\rho)$ and $\lambda_p^{(2)}(\rho)$ are positive. We now recall the definition of $\det \Delta_p$.

The Laplacian Δ_p is a symmetric, positive definite, elliptic operator with pure point spectrum

$$0 < \lambda_1 = \lambda_p(\rho) \leq \lambda_2 \leq \cdots \to +\infty$$

and, writing $e^{-t\Delta_p}(x,y)$ $(x, y \in M)$ for the integral kernel representing the heat kernel on p-forms on M,

$$\mathrm{Tr} e^{-t\Delta_p} = \int_M \mathrm{tr}(e^{-t\Delta_p}(x,x)) dx$$

is convergent for each positive t. We may thus define

$$\log \det \Delta_p = -\frac{d}{ds} \zeta_p(s;\rho)_{|s=0}$$

where the function ζ_p is the unique meromorphic function [9] satisfying

$$\zeta_p(s;\rho) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \mathrm{Tr} e^{-t\Delta_p} dt$$

for $\mathrm{Re}(s)$ sufficiently large.

1.2. Since the trace of the ($\widetilde{M}$-)heat kernel $e^{-t\Delta_k^{(2)}}(x,x)$ on the diagonal is invariant under $\pi_1(M)$ the expression:

$$\mathrm{Tr} e^{-t\Delta_p^{(2)}} = \int_F \mathrm{tr}(e^{-t\Delta_p^{(2)}}(x,x)) dx \quad (t > 0)$$

is independent of the choice of a fundamental domain F for the action of $\pi_1(M)$ on $\widetilde{M}$. Following Lott [6] we define

$$\log \det \Delta_p^{(2)} = -\frac{d}{ds} \zeta_p^{(2)}(s;\rho)_{|s=0}$$

where the function $\zeta_p^{(2)}$ is the unique meromorphic function satisfying

$$\zeta_p^{(2)}(s;\rho) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \mathrm{Tr} e^{-t\Delta_p^{(2)}} dt$$

for $\mathrm{Re}(s)$ sufficiently large.

1.3. We want to investigate the asymptotic behavior of $\det \Delta_p$ when we change the representation ρ. We will make use of the following lemma which follows from [4],[2] as $\pi_1(M)$ acts properly and cocompactly on $\widetilde{M}$.

1.4. **Lemma.** *Let $T \geq 1$. Then there exists a constant $c = c(\widetilde{M}, T)$ such that*

$$||e^{-t\Delta_p^{(2)}}(x,y)|| \leq ct^{-n/2} \exp\left(-\frac{r^2}{4t}\right), \quad 0 < t \leq T,$$

where $x, y \in \widetilde{M}$, r is the distance between x and y, and $|| \cdot ||$ is the norm induced by that on $\Lambda^p_x(E)$.

From this one easily deduces – as in [4] – that for any $t > 0$, we may write[3]

$$e^{-t\Delta_p}(x,x) = e^{-t\Delta_p^{(2)}}(x,x) + O(t^{-n/2}e^{-c/t}). \tag{1.4.1}$$

Here c is a constant which only depends on M. This estimate is uniform for t in a compact subinterval of $[0, +\infty)$. This motivates the following:

1.5. **Definition.** Let (ρ_j) be a sequence of finite dimensional acyclic representations of $\pi_1 M$ and let $m = (m_j)$ be a sequence of positive real numbers. We say the spectrum of $\Delta_p(\rho_j)$ *m-localizes* if

(1) we have:

$$\left|\left| e^{-\frac{t}{m_j}\Delta_p(\rho_j)}(x,x) - e^{-\frac{t}{m_j}\Delta_p^{(2)}(\rho_j)}(x,x) \right|\right| \to 0$$

uniformly on M and on compact t-intervals, and

(2) both sequences

$$\left(e^{-\frac{1}{m_j}\Delta_p(\rho_j)}(x,x)\right)_j \text{ and } \left(e^{-\frac{1}{m_j}\Delta_p^{(2)}(\rho_j)}(x,x)\right)_j$$

remain bounded.

1.6. Now let $(\rho_j)_{j\in\mathbb{N}}$ be a sequence of finite dimensional acyclic representations of $\pi_1 M$ and let $m = (m_j)$ be a sequence of positive real numbers such that the spectrum of $\Delta_p(\rho_j)$ m-localizes.[4]

1.7. **Proposition.** *Suppose that $\lambda_p(\rho_j), \lambda_p^{(2)}(\rho_j) \geq m_j$. Then:*

$$\lim_{j\to+\infty} \frac{1}{\dim \rho_j}\left(\log\det(\Delta_p(\rho_j)) - \log\det(\Delta_p^{(2)}(\rho_j))\right) = 0.$$

Proof. We have $\log\det \Delta_p(\rho_j) = \zeta_p'(0;\rho_j)$ and similarly for the $\Delta_p^{(2)}(\rho_j)$'s. We therefore consider

$$\zeta_p(s;\rho_j) - \zeta_p^{(2)}(s;\rho_j) = \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\left(\mathrm{Tr}e^{-t\Delta_p(\rho_j)} - \mathrm{Tr}e^{-t\Delta_p^{(2)}(\rho_j)}\right) dt.$$

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[2]We thank Jonathan Pfaff for indicating us this reference.

[3]Here and after, given an element $x \in \widetilde{M}$, we abusively set $e^{-t\Delta_p}(x,x) = e^{-t\Delta_p}(\bar{x},\bar{x})$ where $\bar{x}$ is the projection of x in M.

[4]Examples are provided in the next sections.

It follows from (1.4.1) that the integral is holomorphic in s in a half-plane containing 0, so that:

$$
\begin{aligned}
\big(\log\det(\Delta_p(\rho_j)) - &\log\det(\Delta_p^{(2)}(\rho_j))\big) \\
&= \int_0^{+\infty} \left(\mathrm{Tr} e^{-t\Delta_p(\rho_j)} - \mathrm{Tr} e^{-t\Delta_p^{(2)}(\rho_j)} \right) \frac{dt}{t} \\
&= \int_0^{+\infty} \left(\mathrm{Tr} e^{-\frac{t}{m_j}\Delta_p(\rho_j)} - \mathrm{Tr} e^{-\frac{t}{m_j}\Delta_p^{(2)}(\rho_j)} \right) \frac{dt}{t} \\
&= \int_0^{+\infty} \int_F \left(\mathrm{tr} e^{-\frac{t}{m_j}\Delta_p(\rho_j)}(x,x) - \mathrm{tr} e^{-\frac{t}{m_j}\Delta_p^{(2)}(\rho_j)}(x,x) \right) dx \frac{dt}{t}.
\end{aligned}
$$

We split the integral into a sum $\int_0^{+\infty} = \int_0^T + \int_T^{+\infty}$.

First note that for $t \geq 1$, we have

$$
e^{-\frac{t}{m_j}\Delta_p(\rho_j)}(x,x) \leq e^{-\frac{1}{m_j}\Delta_p(\rho_j)}(x,x) \cdot \exp\left(-\lambda_p(\rho_j) \left(\frac{1}{m_j}(t-1) \right) \right).
$$

By hypothesis the sequence $(e^{-\frac{1}{m_j}\Delta_p(\rho_j)}(x,x))$ remains bounded and (since $\lambda_p(\rho_j) \geq m_j$)

$$
\exp\left(-\lambda_p(\rho_j) \left(\frac{1}{m_j}(t-1) \right) \right) \leq \exp(-t+1).
$$

Similar considerations for $\Delta_p^{(2)}(\rho_j)$ imply that the integral

$$
\frac{1}{\dim\rho_j} \int_1^{+\infty} \int_F \left(\mathrm{tr} e^{-\frac{t}{m_j}\Delta_p(\rho_j)}(x,x) - \mathrm{tr} e^{-\frac{t}{m_j}\Delta_p^{(2)}(\rho_j)}(x,x) \right) dx \frac{dt}{t}
$$

is uniformly absolutely convergent. Given a positive real number ε we may therefore choose T such that for all j,

$$
\left| \frac{1}{\dim\rho_j} \int_T^{+\infty} \int_F \left(\mathrm{tr} e^{-\frac{t}{m_j}\Delta_p(\rho_j)}(x,x) - \mathrm{tr} e^{-\frac{t}{m_j}\Delta_p^{(2)}(\rho_j)}(x,x) \right) dx \frac{dt}{t} \right| \leq \varepsilon.
$$

Now, since the spectrum of $\Delta_p(\rho_j)$ m-localizes, we have:

$$
\left| \frac{1}{\dim\rho_j} \int_0^{T} \int_F \left(\mathrm{tr} e^{-\frac{t}{m_j}\Delta_p(\rho_j)}(x,x) - \mathrm{tr} e^{-\frac{t}{m_j}\Delta_p^{(2)}(\rho_j)}(x,x) \right) dx \frac{dt}{t} \right| \to 0
$$

as j tends to infinity. The proposition therefore follows. □

1.8. **Analytic torsion.** Assume that all $\lambda_p(\rho)$ are positive. We denote by $T_M(\rho)$ the analytic torsion of (M, E_ρ), defined as:

$$
\log T_M(\rho) = \frac{1}{2} \sum_{k\geq 0} (-1)^{k+1} k \ \log\det\Delta_k.
$$

In the next sections we derive applications of proposition 1.7 to asymptotic results of analytic torsion.

2. Growth of torsion in finite covers

We consider homogeneous manifolds. Let $\mathbf{G}$ be a semisimple algebraic group over $\mathbb{R}$; let G be the connected component of $\mathbf{G}(\mathbb{R})$.

Let $K \subset G$ be a maximal compact subgroup, with Lie algebra $\mathfrak{k}$. Since G is connected, so also is K. We denote by S the global Riemannian symmetric space G/K. Let Γ be a cocompact torsion-free subgroup of G and let $M = \Gamma\backslash S$.

Let U be a maximal compact subgroup of $G_{\mathbb{C}}$. Let ρ be an irreducible representation of U; it extends to a unique holomorphic representation of $G_{\mathbb{C}}$ on a complex vector space E_ρ, or indeed an algebraic representation of $\mathbf{G}$.

There is, up to scaling, one U-invariant Hermitian metric on E_ρ. We fix an inner product $(-,-)_E$ in this class.

Set V to be the vector bundle on X induced by ρ (i.e., the quotient of the total space $E_\rho \times S$ by the Γ-action).

2.1. **L^2-torsion.** The trace of the (S-)heat kernel $e^{-t\Delta_k^{(2)}}(x,x)$ on the diagonal is independent of x, because it is invariant under G. We prove in [1] that the integral

$$\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\mathrm{tr}\ e^{-t\Delta_k^{(2)}}(x,x)dt$$

is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large and extends to a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s=0$.

Define $t_S^{(2)}(\rho)$ via

$$t_S^{(2)}(\rho) = \frac{1}{2}\sum_{k\geq 0}(-1)^k k\left(\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\mathrm{tr}\ e^{-t\Delta_k^{(2)}}(x,x)dt\right). \tag{2.1.1}$$

(The product of $t_S^{(2)}(\rho)$ with the volume of M is the L^2-analytic torsion of M.) It is possible to compute $t_S^{(2)}(\rho)$ in a completely explicit fashion – it is an explicit quantity depending on ρ, see [1].

2.2. In this section we consider the situation where ρ is fixed but we let vary Γ: Let (Γ_j) is a sequence of finite index subgroups of Γ such that

$$d_j = \inf_{\substack{\in\Gamma_j \\ \gamma\neq 1}}\inf_{x\in M} d(x,\gamma x)$$

tends to infinity with j. We let ρ_j be the representation obtained by taking the diagonal action of G on the tensor product $\mathbb{C}[\Gamma/\Gamma_j]\otimes_{\mathbb{C}} E$ where G acts through the left regular representation on $\mathbb{C}[\Gamma/\Gamma_j]$ and acts by ρ on E.

We note that:

$$\begin{aligned}
\mathrm{Tr}e^{-t\Delta_p(\rho_j)} &= \int_{\Gamma\backslash S}\mathrm{tr}e^{-t\Delta_p(\rho_j)}(x,x)dx \\
&= \int_{\Gamma\backslash(S\times(\Gamma/\Gamma_j))}\mathrm{tr}e^{-t\Delta_p(\rho_j)}(x,x)dx \\
&= \int_{\Gamma_i\backslash S}\mathrm{tr}e^{-t\Delta_p^j(\rho)}(x,x)dx \\
&= \mathrm{Tr}e^{-t\Delta_p^j(\rho)}.
\end{aligned}$$

And similarly:

$$\mathrm{Tr} e^{-t\Delta_p^{(2)}(\rho_j)} = [\Gamma : \Gamma_j] \mathrm{Tr} e^{-t\Delta_p^{(2)}(\rho)}.$$

We let $\lambda_p^j(\rho)$ denote the lowest eigenvalue of $\Delta_p(\rho_j)$ or equivalently of $\Delta_p(\rho)$ on M_j. Proposition 1.7 then implies:

2.3. **Proposition.** *Suppose that there exists $\varepsilon > 0$ such that for all j and p, we have $\lambda_p^j(\rho) \geq \varepsilon$. Then:*

$$\lim_j \frac{\log T_{M_j}(\rho)}{[\Gamma : \Gamma_j]} = t_M^{(2)}(\rho)\mathrm{vol}(M).$$

Proof. We take $(m_j) = (\varepsilon)$ (constant sequence). Using lemma 1.4 and (1.4.1) one easily checks that (2) of definition 1.5 holds and (1) follows fom the fact that d_j tends to infinity. □

Many examples where this proposition applies are given in [1]. We note in particular that it applies to hyperbolic 3-manifolds with the standard $\mathbb{C}^2$ representation, see [5]. Given an orientable hyperbolic 3-manifold M we consider the discrete and faithful representation α_{can} of $\pi_1(M)$ in $\mathrm{SL}_2(\mathbb{C})$. The corresponding twisted chain complex

$$C_*(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{C}^2$$

is acyclic and we denote by $\tau(M, \alpha_{\mathrm{can}}) \in \mathbb{R}^*$ the corresponding torsion.

2.4. **Corollary.** *Let M be a closed hyperbolic 3-manifold . Let $\{M_i\}_{i\in\mathbb{N}}$ be a nested collection of finite covers such that $\cap_{i\in\mathbb{N}}\pi_1(M_i) = 1$. Then:*

$$\lim_{i\to+\infty} \frac{\log|\tau(M_i, \alpha_{\mathrm{can}})|}{\mathrm{vol}(M_i)} = -\frac{11}{12\pi}.$$

3. Asymptotic of twisted torsion

We now consider the case where Γ is fixed and we vary ρ.

3.1. **Notation.** We continue with notations as in the preceeding paragraph. If H is a real Lie group, we will write $\mathfrak{h}$ for the *complexification* of its real Lie algebra, and $\mathfrak{h}_{\mathbb{R}}$ for the Lie algebra of H.

3.1.1. *Groups and subgroups.* We have already defined $K \subset G$ and U a compact form of G. Note that we may identify the complexified Lie algebra $\mathfrak{u}$ of U and the complexified Lie algebra of $\mathfrak{g}$. Let $S = G/K$ be the Riemannian symmetric space associated to G and $S^c = U/K$ be its compact dual. Let Θ be the Cartan involution of G fixing K.

Fix a maximal torus $T_f \subset K$ with complexified Lie algebra $\mathfrak{b}$. Extend it to a Θ-stable maximal torus $\mathfrak{t}_U = \mathfrak{b} \oplus \mathfrak{a}_0 \subset \mathfrak{u}$, where $\mathfrak{a}_0 \subset \mathfrak{p}$ is the complexification of an abelian subspace $\mathfrak{a}_{0\mathbb{R}} \subset \mathfrak{p}_{\mathbb{R}}$. Therefore, $\mathfrak{b}\oplus\mathfrak{a}_0$ is a "fundamental Cartan subalgebra," i.e. one with maximal compact part.

Let $T_U \subset U, W_U \subset \mathrm{Aut}(\mathfrak{t}_U)$ be the maximal tori and Weyl groups that correspond to $\mathfrak{t}_U \subset \mathfrak{g}$.

3.1.2. *Systems of positive roots.* We choose a system $\Delta^+ = \Delta^+(\mathfrak{t}_U, \mathfrak{u})$ of positive roots for the action of $\mathfrak{t}_U$ on $\mathfrak{u}$.

Let $\rho_U \in \mathfrak{t}_U^*$ be the half-sum of positive roots in Δ^+.

3.1.3. *Representations.* Let ρ_λ be an irreducible representation of U with (dominant) highest weight $\lambda \in \mathfrak{t}_U^*$; it extends to a unique holomorphic representation of $G_{\mathbb{C}}$, which we also denote ρ_λ.

For each $w \in W_U$, let μ_w be the restriction of $w(\rho_U + \lambda)$ to $\mathfrak{b}$.

3.2. **Small time estimate for the heat kernel.** We first note that the proof of [1, Lemma 3.8] implies that in lemma 1.4 one can write

$$||e^{-t\Delta_p^{(2)}}(x,y)|| \leq ce^{-t\Lambda_\rho}t^{-n/2}\exp\left(-\frac{r^2}{5t}\right), \quad 0 < t \leq T,$$

with c independent of ρ and $\Lambda_\rho = |\lambda + \rho_U|^2 - |\rho_U|^2 \geq 0$.

Now consider a sequence (ρ_j) of irreducible representations of U with (dominant) highest weight λ_j. Proposition 1.7 implies the following:

3.3. **Proposition.** *Assume that*

$$\min_{w \in W_U}(|\lambda_j + \rho_U|^2 - |\mu_w^j|^2) \overset{j\to+\infty}{\longrightarrow} +\infty.$$

Then:

$$\lim_{j\to+\infty} \frac{\log(T_M(\rho_j)) - t_S^{(2)}(\rho_j)\mathrm{vol}(M)}{\dim \rho_j} = 0.$$

Proof. Let $m_j = \min_{w\in W_U}(|\lambda_j + \rho_U|^2 - |\mu_w^j|^2)$. Then $b_j \to +\infty$ with j and (1) of definition 1.5 follows from (1.4.1). Point (2) follows from §3.2. It finally follows from [1, Lemma 4.1 and §5.7] that for any p the lowest eigenvalues of $\Delta_p(\rho_j)$ and $\Delta_p(\rho_j)^{(2)}$ are bigger than m_j. Proposition 1.7 therefore applies and the theorem follows. □

This theorem applies to many examples, we compute several of them following [1, §5.9] in the next section. Before we apply it to $G = \mathrm{SL}_2(\mathbb{C})$ and recover a recent result of Mueller [8].

3.4. Let $G = \mathrm{SL}_2(\mathbb{C})$ and let $\rho_m = \mathrm{Sym}^m$ be the m-th symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$. It follows from [1, §5.9.1] that

$$\min_{w\in W_U}(|\lambda_m + \rho_U|^2 - |\mu_w^m|^2) = m^2$$

and

$$t_{\mathbb{H}^3}^{(2)}(\rho_m) = -\frac{1}{4\pi}m^2 - \frac{1}{2\pi}m - \frac{1}{6\pi}.$$

Theorem 3.3 therefore implies the following generalization of the main theorem of [8].

3.5. **Corollary.** *Let M be a closed, oriented hyperbolic 3-manifold. Then we have:*

$$-\log T_M(\rho_m) = \frac{\mathrm{vol}(M)}{4\pi}m^2 + O(m)$$

as $m \to +\infty$.

4. More examples

In this section we provide some more examples. After we had written a first draft of this short note, the paper of Mueller and Pfaff [7] appeared (see also Bismut-Ma-Zhang [2]). They obtain similar results. Their method is similar in nature, note however that the use of the trace formula allows them to deal with finite volume (non-compact) hyperbolic manifolds in a subsequent paper.

4.1. **Hyperbolic manifolds.** There is a natural generalisation of corollary 3.5 to higher odd dimensional hyperbolic manifolds.

4.1.1. Indeed: consider the case where $G = \mathrm{SO}(2n-1,1)$. In this case $U \cong \mathrm{SO}_{2n}(\mathbb{R})$ and $S = \mathbb{H}^{2n-1}$.

In the notation of [3], we may choose a Killing-orthogonal basis ε_i for $\mathfrak{t}_U^*$ such that:

(1) The positive roots are those roots $\varepsilon_k \pm \varepsilon_l$ with $1 \leq k < l \leq n$;
(2) $\mathfrak{a}$ is the common kernel of $\varepsilon_2, \ldots, \varepsilon_{n-1}$;
(3) $\mathfrak{b}$ is the kernel of ε_1, and the positive roots for $\mathfrak{b}$ on $M_f \cap K$ are $\varepsilon_j \pm \varepsilon_k$ $(1 < j < k)$.
(4) The positive roots nonvanishing on $\mathfrak{a}$ are $\varepsilon_1 \pm \varepsilon_l$ $(1 < l \leq n)$, and thus $\alpha_0 = \varepsilon_1$ gives the unique positive restricted root $\mathfrak{a}_{0,\mathbb{R}} \to \mathbb{R}$.

The representations of U are parametrized by a highest weight $\lambda = (\lambda_1, \ldots, \lambda_n) = \lambda_1\varepsilon_1 + \ldots + \lambda_n\varepsilon_n$ such that λ is dominant (i.e. $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|$) and integral (i.e. every $\lambda_i \in \mathbb{Z}$).

We have $\rho_U + \lambda = (n-1+\lambda_1, n-1+\lambda_2, \ldots, \lambda_n)$. We may assume that $\lambda_n \geq 0$ and for convenience we will rewrite

$$\rho_U + \lambda = (a_{n-1}, \ldots, a_0).$$

Note that (a_j) is a strictly increasing sequence of nonnegative integers and that

$$\min_{w \in W_U} (|\rho_U + \lambda|^2 - |\mu_w|^2) = a_0^2.$$

Now consider a sequence ρ_m of representations and denote by $(a_{n-1}^m, \ldots, a_0^m)$ the corresponding quantities. We assume that $(a_0^m)^2 \to +\infty$ with m and that each sequence $|a_k^m - a_{k-1}^m|$ is bounded. Set

$$E_m = \prod_{0 \leq i < j \leq n-1} ((a_j^m)^2 - (a_i^m)^2), \quad F = \prod_{0 \leq i < j \leq n} (j^2 - i^2).$$

Let Q_k^m be the unique even polynomial of degree $\leq 2n-2$ which satisfies

$$Q_k^m(\pm a_j^m) = \begin{cases} 0 & \text{if } j < k \\ 1 & \text{if } j \geq k. \end{cases}$$

It follows from [1, §5.9.1] that:

$$t^{(2)}_{\mathbb{H}^{2n-1}}(\rho_m) = (-1)^{n-1} \frac{(n-1)!}{2\pi^{n-1}} \frac{E_m}{F} \sum_{k=0}^{n-1} \int_{a_{k-1}^m}^{a_k^m} Q_k^m(t)dt. \tag{4.1.1}$$

Here we set $a_{-1}^m = 0$. Each integral $\int_{a_{k-1}^m}^{a_k^m} Q_k(t)^m dt$ $(k > 1)$ in (4.1.1) is bounded since Q_k^m increase from 0 to 1 and $|a_k^m - a_{k-1}^m|$ is bounded. And $\int_0^{a_0^m} Q_0(t)^m dt = a_0^m$ since $Q_0^m \equiv 1$. Finally:

$$t^{(2)}_{\mathbb{H}^{2n-1}}(\rho_m) = (-1)^{n-1} \frac{(n-1)!}{2\pi^{n-1}} \frac{E_m}{F} \big(a_0^m + O(1)\big).$$

This applies to the representation ρ_m of weight $(m, \ldots, m)$ which corresponds to the m-th symmetric power $\mathrm{Sym}^m(V^+)$ of the standard representation of G in the irreducible subspace $V^+ \subset \wedge^m \mathbb{C}^{2m}$. We get the following:

4.2. **Corollary.** *Let M be a closed, oriented hyperbolic $(2n-1)$-manifold. Then we have:*

$$\log T_M(\rho_m) = (-1)^{n-1} \frac{(n-1)! 2^{\frac{n(n-1)}{2}} \mathrm{vol}(M)}{2\pi^{n-1} \prod_{0 \le i < j \le n-1} (j+i)} m^{\frac{n(n-1)}{2}+1} + O(m^{\frac{n(n-1)}{2}})$$

as $m \to +\infty$.

Proof. This follows from the fact that

$$E = \prod_{0 \le i < j \le n-1} ((m+j)^2 - (m+i)^2)$$

and

$$F = \prod_{0 \le i < j \le n-1} (j^2 - i^2)$$

so that

$$\frac{E}{F} = \prod_{0 \le i < j \le n-1} \frac{2n+j+i}{j+i}.$$

□

We finally deal with the case of $G = \mathrm{SL}_3(\mathbb{R})$.

4.3. **The case of $\mathrm{SL}_3(\mathbb{R})$.** In this case $U \cong \mathrm{SU}_3$, $K \cong \mathrm{SO}_3$, $M_f \cong \{g \in \mathrm{GL}_2(\mathbb{R}) : \det g = \pm 1\}$, and $K_f \cong \mathrm{O}_2(\mathbb{R})$.

4.4. Fix an element $g \in \mathrm{SU}_3$ conjugating $\mathfrak{t}_U$ into diagonal matrices; let ε_j $(1 \le j \le 3)$ be the pull-back, by g, of the coordinate functionals. Thus $\sum \varepsilon_j = 0$; moreover, we may choose g such that:

(1) The positive roots are $\varepsilon_i - \varepsilon_j$, with $i < j$;
(2) $\rho_U = \varepsilon_1 - \varepsilon_3$;
(3) $\mathfrak{b}$ is identified with the kernel of $\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3$;
(4) $\alpha_0 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3)$ is the unique restricted root $\mathfrak{a}_{0,\mathbb{R}} \to \mathbb{R}$;
(5) The Weyl group S_3 acts by permuting the ε_j.

Now let $p \ge q \ge r \in \mathbb{Z}$ and set $\lambda = p\varepsilon_1 + q\varepsilon_2 + r\varepsilon_3$. Put

$$A_1 = \frac{1}{2}(p+1-q), \quad A_2 = \frac{1}{2}(p-r+2), \quad \text{and} \quad A_3 = \frac{1}{2}(q-r+1).$$

$$C_1 = \frac{1}{3}(p+q-2r+3), \quad C_2 = \frac{1}{3}(p+r-2q), \quad \text{and} \quad C_3 = \frac{1}{3}(2p-q-r+3).$$

4.4.1. *Normalisation.* Fix an invariant metric on $S = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$ induced from the trace form of the standard representation of $\mathfrak{sl}_3(\mathbb{R})$. It induces a Riemannian volume form dx on S and on the compact dual S^c as well. And we deduce from [1, §5.9.2] that:

$$t_S^{(2)}(\rho) = \frac{\pi}{2\mathrm{vol}(S^c)} \left(A_1 A_3 C_1 C_3 + A_2 |C_2| \begin{cases} A_3 C_3, & C_2 \ge 0 \\ A_1 C_1, & C_2 \le 0. \end{cases} \right)$$

Theorem 3.3 applies as soon as $|C_2^j| \to +\infty$.

4.4.2. *An example.* We apply the previous discussion to the representation ρ_m of weight $(m, 0, 0)$ which corresponds to the m-th symmetric power of the standard representation of G in $\mathbb{C}^3$. We get the following:

4.5. **Corollary.** *Let M be a closed, oriented manifold which is the quotient of the symmetric space associated to $\mathrm{SL}_3(\mathbb{R})$. Then we have:*

$$\log T_M(\rho_m) = \frac{2\pi \mathrm{vol}(M)}{9\mathrm{vol}(S^c)} m^3 + O(m^2)$$

as $m \to +\infty$.

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