l^1-Measure equivalence rigidity of hyperbolic lattices

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l^p-Measure equivalence

Measure equivalence with l^p-condition

A ME-coupling (Ω, μ) of Γ and Λ is a measure space with a μ-preserving action of $\Gamma \times \Lambda$ such that Γ, Λ both have a μ-finite fundamental domain. If Γ and Λ admit a ME-coupling with l^p-integrable cocycles w.r.t. some fundamental domains then we call them l^p-measure equivalent.

l^p-coycles

A measurable cocycle $\alpha : \Gamma \times (X, \mu) \to \Lambda$ is l^p-integrable if for every $\gamma \in \Gamma$

$$\int_X l(\alpha(\gamma, x))^p d\mu(x) < \infty,$$

where $l : \Lambda \to \mathbb{N}$ is the length function for some word metric on Λ.

- l^p-ME interpolates between $p = \infty$ ($\Rightarrow$ QI) and $p = 0$ (=ME).
- l^p-ME is an equivalence relation on groups.

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Rigidity result for hyperbolic lattices

Theorem (informal)

Let Γ be a lattice in $G = \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 3$. Then any l^1-ME-coupling of Γ with another group basically comes from the standard example of lattices in G or atomic couplings of commensurable groups.

- The standard coupling of hyperbolic lattices is l^1-integrable.
- A corresponding rigidity result for orbit equivalence (OE) can be formulated.
- Analogous rigidity results (without any l^1-integrability condition) for lattices in higher rank Lie groups hold true [Furman, 2000].
- Lack of rigidity for $n = 2$: $\mathbb{Z}^2 * \mathbb{Z}^2$ OE to $\mathbb{Z} * \mathbb{Z}$.

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Precise rigidity result – ME-version

Theorem (Bader-Furman-S.)

Let Γ be a lattice in $G = \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 3$. Let (Ω, μ) be an ergodic, l^1-integrable ME-coupling with another group Λ.

Then the following holds:

a) *There exists a homomorphism $\rho : \Lambda \to G$ with finite kernel and image being a lattice in G.*

b) *There exists a $\Gamma \times \Lambda$-equivariant measurable map $\phi : \Omega \to G$; the push-forward measure $\phi_* \mu$ is the Haar measure corresponding either*
   i) *to G,*
   ii) *or to its index two subgroup $G^0 = \mathrm{Isom}_+(\mathbb{H}^n)$,*
   iii) *or to a lattice $\Gamma' < G$.*

   In the latter case, Γ' contains Γ and $\rho(\Lambda)$ as subgroups of finite index.

Classical Mostow rigidity

Theorem (Mostow rigidity – Lie-theoretic version)

Any isomorphism $\Gamma \to \Lambda$ *of lattices in* $G = \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 3$, *extends to an automorphism of* G.

Theorem (Mostow rigidity – topological version)

Let M *and* N *be closed hyperbolic* n*-dimensional manifolds. Then any homotopy equivalence* $M \to N$ *is homotopic to an isometry.*

topological version $\Rightarrow$ Lie-theoretic version:

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$$\begin{array}{ccc} \widetilde{M} & \dashrightarrow & \widetilde{N} \\ \uparrow & & \uparrow \\ \Gamma & \longrightarrow & \Lambda \end{array}$$

Induction over skeleta

$$\begin{array}{ccccc} \coprod_{S_i} \Gamma \times \partial\Delta^i & \longrightarrow & \widetilde{M}^{(i-1)} & & \\ \downarrow & & \downarrow & \searrow & \\ \coprod_{S_i} \Gamma \times \Delta^i & \longrightarrow & \widetilde{M}^{(i)} & \dashrightarrow & \widetilde{N} \end{array}$$

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Thurston's proof of (topological) Mostow rigidity

Proof for closed manifolds

Step 1) $f : M \xrightarrow{\cong} N \Rightarrow \|M\| = \|N\| \Rightarrow \mathrm{vol}(M) = \mathrm{vol}(N)$ [Gromov-Thurston].

Step 2) $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ is a quasi-isometry, thus induces a homeomorphism $\partial_\infty \tilde{f} : \partial_\infty \mathbb{H}^n \xrightarrow{\cong} \partial_\infty \mathbb{H}^n$.

Step 3) Regular, ideal n-simplices are exactly the geodesic n-simplices with maximal volume [Haagerup-Munkholm].
$\partial_\infty \tilde{f}$ preserves regular, ideal simplices.

Step 4) Hyperbolic geometry: $\partial_\infty \tilde{f}$ induced by an isometry.

Modification for finite volume manifolds

Only from volume considerations, Thurston constructs a measurable $\partial_\infty \tilde{f}$ that preserves regular, ideal n-simplices almost everywhere.

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Reduction of main theorem to cocycle Mostow rigidity I

Theorem (adapted from Furman's earlier work)

Let Γ *be a lattice in* $G = \mathrm{Isom}(\mathbb{H}^n)$, *and* Λ *be an arbitrary group ME to* Γ *via the coupling* (Ω, m). *Let* $(\Sigma, n) = (\Omega, m) \times_\Lambda (\Omega^{\mathrm{op}}, m)$ *be the corresponding self-coupling of* Γ. *Assume that there exists a measurable* $\Gamma \times \Gamma$*-equivariant map* $\Phi : \Sigma \to G$ *("untwisting map"), i.e. n-a.e.*

$$\Phi([\gamma x, \gamma' y]) = \gamma \Phi([x, y]) \gamma'^{-1} \qquad (\gamma, \gamma' \in \Gamma).$$

Then there exist measurable maps $f : \Omega \to G$ *and a homomorphism* $\rho : \Lambda \to G$ *so that*

$$f((\gamma, \lambda)x) = \gamma f(x) \rho(\lambda)^{-1}.$$

Then elementary observations (for lattice image) and an application of Ratner's theorems (for identifying $\Phi_* n$) eventually yield the main theorem.

. . . How do we get the untwisting map?

Reduction of main theorem to cocycle Mostow rigidity II

Setting

- Let $X \subset \Sigma$ be a common fundamental domain of both copies of Γ, and $\alpha : \Gamma \times X \to X$ be the corresponding OE-cocycle.
- We may assume that $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is co-compact. Let $M = \Gamma \backslash \mathbb{H}^n$.

Proof of main theorem – outline

Step 1) Extend $\alpha : X \to \mathrm{map}(\Gamma, \Gamma)$ to a α-equivariant, measurable map $\psi : X \to \mathrm{map}(\widetilde{M}, \widetilde{M})$.

Step 2) Show that ψ induces a measurable, α-equivariant map $\partial_\infty \psi : X \to \mathcal{M}(\partial \mathbb{H}^n, \partial \mathbb{H}^n)$ that preserves regular, ideal n-simplices.

Step 3) Hyperbolic geometry $\Rightarrow$ $\partial_\infty \psi$ comes from of a α-equivariant map $\phi : X \to \mathrm{Isom}(\widetilde{M}) = G$ (cocycle Mostow rigidity).

Step 4) ϕ is a coboundary for α; thus we can also untwist Σ.

A crucial step in the proof – controlling volume

Lemma

For any geodesic simplex σ with $\mathrm{vol}(\sigma) \approx v_{max}$ we have

$$\int_X \int_{\Gamma\backslash G} \mathrm{vol}(\psi_x(g\sigma)) d\mu_X(x) d\mu_{\Gamma\backslash G}(g) \approx \mathrm{vol}(\sigma).$$

Volume and degree 1 maps

Let $f : M \to M$ be a degree 1 map. Let $c = \sum a_i \sigma_i$ be an n-cycle. Then:

$$\sum a_i \mathrm{vol}^{\mathrm{or}}(\sigma_i) = \sum a_i \mathrm{vol}^{\mathrm{or}}(f(\sigma)).$$

- Find suitable homology theories for our situation.
- Show that $\psi : X \to \mathrm{map}(\widetilde{M}, \widetilde{M})$ is of degree 1.
- View left side of lemma as the evaluation of a homology class at the volume form.

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l^1-homology and induced maps

Maps induced by $\alpha : \Gamma \times X \to \Gamma$ and $\psi : X \to \mathrm{map}(\widetilde{M}, \widetilde{M})$

$$\begin{array}{ccc} \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*(\Gamma) & \hookrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{geo}}(\widetilde{M}) \\ \Big\downarrow {\scriptstyle C_*(\alpha)} & & \Big\downarrow {\scriptstyle C_*^{\mathrm{geo}}(\phi)} \\ \overline{L^1(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\Gamma)}^1 & \hookrightarrow & \overline{L^1(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{geo}}(\widetilde{M})}^1 \end{array}$$

Remarks

- $L^1(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{geo}}(\widetilde{M}) = \bigoplus_F L^1(X;\mathbb{Z})$
- Vertical maps are inclusions of orbits.
- $C_0(\alpha)(1 \otimes \gamma) = \sum \chi_{X_i} \otimes \gamma_i$ where $\alpha(x,\gamma) = \gamma_i$ constant on $x \in X_i$.

A new deformation-rigidity phenomenon

Integrality, Poincare duality, simplicial volume

We have by Poincare duality and ergodicity

$$H_n(L^1(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{geo}}(\widetilde{M})) \cong H^0(\widetilde{M}; L^1(X;\mathbb{Z})) = L^1(X;\mathbb{Z})^\Gamma \cong \mathbb{Z}.$$

Since the simplicial volume of M is > 0 every Cauchy sequence of cycles in $L^1(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n^{\mathrm{geo}}(\widetilde{M})$ is eventually constant!

Sobolev homology and l^1-condition

Under l^1-integrability, we show that $H_n(\phi)$ already lands in

$$H_n(L^1(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{geo}}(\widetilde{M})) \subset H_n\left(\overline{L^1(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{geo}}(\widetilde{M})}^1\right)$$

For this we use a new tool (Sobolev homology) and the ability to subdivide geodesic simplices in negative curvature very efficiently.

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