VOLUME AND L^2-BETTI NUMBERS OF ASPHERICAL MANIFOLDS

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Abstract. We give a leisurely account of the relationship between volume and L^2-Betti numbers on closed, aspherical manifolds based on the results in [5] – albeit with a different point of view. This paper grew out of a talk presented at the first colloquium of the Courant Center in Göttingen in October 2007.

1. Review of L^2-Betti numbers

The L^2-Betti numbers of a closed Riemannian manifold, as introduced by Michael Atiyah, are analytical invariants of the long-time behavior of the heat kernel of the Laplacians of forms on the universal cover. We give a very brief review of these invariants; for extensive information the reader is referred to the standard reference [3].

Let $\widetilde{X} \to X$ be the universal cover of a compact Riemannian manifold, and let $\mathscr{F} \subset \widetilde{X}$ be a $\pi_1(X)$-fundamental domain. Then Michael Atiyah defines the i-th L^2-Betti number in terms of the heat kernel on $\widetilde{X}$ as

$$b_i^{(2)}(X) = \lim_{t\to\infty} \int_{\mathscr{F}} \operatorname{tr}_{\mathbb{C}} e^{-t\Delta_i}(x,x)dvol(x).$$

Subsequently, simplicial and homological definitions of L^2-Betti numbers were developed by Dodziuk, Farber, and Lück. An important consequence of the equivalence of these definitions is the homotopy invariance of L^2-Betti numbers.

Lück's definition is based on a dimension function $\dim_{\mathcal{A}}(M)$ for arbitrary modules M over a finite von Neumann algebra $\mathcal{A}$ with trace $\operatorname{tr} : \mathcal{A} \to \mathbb{C}$. For example, one has $\dim_{\mathcal{A}}(\mathcal{A}p) = \operatorname{tr}(p)$. Lück proceeds then to define $b_i^{(2)}(X)$ for an arbitrary space X with $\Gamma = \pi_1(X)$ as

$$b_i^{(2)}(X) = \dim_{L(\Gamma)} H_i\big(L(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{X})\big) \in [0,\infty] \tag{1.1}$$

where $L(\Gamma)$ is the group von Neumann algebra of Γ. Some of the most fundamental properties of L^2-Betti numbers are:

- $\pi_1(X)$ finite $\Rightarrow b_i^{(2)}(X) = b_i(\widetilde{X})/|\pi_1(X)|$
- $\sum_{i\geq 0}(-1)^i b_i^{(2)}(X) = \chi(X) = \sum_{i\geq 0}(-1)^i b_i(X)$.
- $\bar{X} \to X$ d-sheeted cover $\Rightarrow b_i^{(2)}(\bar{X}) = d \cdot b_i^{(2)}(X)$.
- If X is aspherical and $\pi_1(X)$ amenable then $b_i^{(2)}(X) = 0$.
- If X is a $2n$-dimensional hyperbolic manifold then $b_i^{(2)}(X) > 0$ if and only if $i = n$.

2. Theorems relating volume and L^2-Betti numbers

Assumption 2.1. *Throughout this section, let M be an n-dimensional, closed, aspherical manifold.*

The inequality of Theorem 2.2 is stated by Mikhail Gromov [2, Section 5.33 on p. 297] along with an idea[1] which he attributes to Alain Connes. We provide the first complete proof of that inequality [5, Corollary to Theorem A]. The rigorous implementation of Gromov's idea uses tools and ideas from Damien Gaboriau's theory of L^2-Betti numbers of measured equivalence relations and spaces with groupoid actions of such.

Theorem 2.2. *If (M, g) has a lower Ricci curvature bound $\mathrm{Ricci}(M, g) \geq -(n-1)g$, then*

$$b_i^{(2)}(M) \leq \mathrm{const}_n \,\mathrm{vol}(M, g) \qquad \textit{for every } i \geq 0.$$

The *minimal volume* of a smooth manifold N is defined as the infimum of volumes of complete metrics on N whose sectional curvature is pinched between -1 and 1. We obtain the following

Corollary 2.3 (Minimal volume estimate)**.**

$$b_i^{(2)}(M) \leq \mathrm{const}_n \,\mathrm{minvol}(M).$$

The following theorem [5, Theorem B] is a generalization of a well-known vanishing result of Jeff Cheeger and Mikhail Gromov. Its connection to volume becomes apparent through its corollary.

Theorem 2.4. *If M is covered by open, amenable sets such that every point belongs to at most n sets, then*

$$b_i^{(2)}(M) = 0 \qquad \textit{for every } i \geq 0.$$

Here a subset $U \subset M$ is called *amenable* if $\pi_1(U)$ maps to an amenable subgroup of $\pi_1(M)$. There is also a version of this theorem for arbitrary spaces [5, Theorem C]. The following corollary is a non-trivial implication of the theorem above and work of Mikhail Gromov [1, Section 3.4] where he constructs amenable coverings in the presence of small volume.

Corollary 2.5. *There is a constant $\epsilon_n > 0$ only depending on n such that*

$$\mathrm{minvol}(M) < \epsilon_n \Rightarrow b_i^{(2)}(M) = 0 \qquad \textit{for every } i \geq 0.$$

The results above are analogs of well-known theorems by Mikhael Gromov where *L^2-Betti numbers* are replaced by *simplicial volume*. Note however that the assumption of asphericity is crucial here unlike in the case of the simplicial volume.

3. Idea of proof of the main theorem

We describe some ideas involved in the proof of Theorems 2.2 and 2.4. In Subsection 3.1 we describe a general technique of bounding (L^2-)Betti numbers by constructing suitable equivariant coverings on the universal cover. Since the assumptions of our theorems are too weak to garantuee the existence of such covers we need substantially modify this technique; the new tool runs under the name

[1] We refer to this idea as *randomization*.

randomization, and it is explained in Subsection 3.2. A full proof based on randomization is rather long and complicated; we explain instead an instructive toy example in Subsection 3.4. A crucial property of L^2-Betti numbers is described in Subsection 3.3. We conclude this sketch of proof in Subsection 3.5 with some remarks about other ingredients.

Throughout the section, we refer to Assumption 2.1.

3.1. **How to bound L^2-Betti numbers by equivariant coverings in general.** Let $\Gamma = \pi_1(M)$. Suppose we construct, under a certain geometrical assumption, a Γ-equivariant open covering $\mathcal{U}$ of the universal cover $\widetilde{M}$. Let us say that $\mathcal{U} = \{U_i\}_{i\in I}$ is indexed by a free Γ-set set I, and we have $\gamma U_i = U_{\gamma i}$. By a standard argument (partition of unity) one obtains a Γ-equivariant map f from $\widetilde{M}$ to the nerve of $\mathcal{U}$. The nerve is embedded in the full simplicial complex with index set I which we denote by $\Delta(I)$. Let

$$\Omega = \operatorname{map}\bigl(\widetilde{M}, \Delta(I)\bigr)$$

be the space of continuous maps with the natural Γ-action. We may view f as an element in Ω^Γ, the subspace of Ω consisting of Γ-equivariant maps. Next we argue that both the i-th Betti number and the L^2-Betti number are bounded from above by the number of equivariant i-simplices hit by $f(\widetilde{M})$.

Let $\mathscr{F}_i$ be a set of Γ-representatives of the i-skeleton $\Delta(I)^{(i)}$. For any $g \in \Omega$, let $C_i(g) \in \mathbb{N}$ be the number of i-simplices in $\mathscr{F}_i$ hit by $f(\widetilde{M})$. We think of C_i as a function

$$C_i : \Omega \to \mathbb{Z}.$$

Since $\widetilde{M}$ is contractible, M is a model of the classifying space $B\Gamma$, and the universal property of $E\Gamma$, the universal cover of $B\Gamma$, implies that there is an equivariant homotopy retract

$$\begin{tikzcd} \widetilde{M} \arrow[r, "f"'] & \Delta(I) \arrow[l, bend right] \end{tikzcd}.$$

Using the fact that the i-th L^2-Betti number is bounded by the number of equivariant i-simplices and the fact that the L^2-Betti number is some sort of dimension (with nice properties) of a certain homology module (see (1.1)), we easily obtain that

$$b_i^{(2)}(M) \le C_i(f).$$

By going to Γ-quotients we also obtain the same estimate for the usual Betti numbers. By Poincare duality it is actually enough to control $C_n(f)$, and we have

$$b_i(M), b_i^{(2)}(M) \le \operatorname{const}_n C_n(f) \tag{3.1}$$

for a constant const_n only depending on n. This follows from [3, Example 14.28 on p. 498] since the fundamental class of M can be written as a sum of at most $C_n(f)$ singular simplices.

So to get a good bound on $b_i^{(2)}(M)$, we should find an equivariant cover $\mathcal{U}$ such that for the resulting map f to the nerve the quantity $C_n(f)$ is rather small.

3.2. **Randomization.** One directly sees the limitations of the above technique. The trivial estimate $C_n(f) \ge 1$ for any map $f \in \Omega$ prevents us from proving the vanishing of the L^2-Betti numbers. In particular, we cannot hope to prove Theorems 2.2 and 2.4 using it.

Next we phrase an idea of Mikhail Gromov (attributed to Alain Connes) in probabilistic terms that modifies the above technique.

By changing the point of view a bit, we regard a map $f \in \Omega^\Gamma$ that we sought to construct before as a Γ-invariant point measure on the Borel space Ω. Instead of trying to find a point measure f with small $C_n(f)$, Gromov suggests to look for Γ-invariant probability measures μ on Ω such that the *expected value*

$$\mathbb{E}_{(\Omega,\mu)}(C_n) = \int_\Omega C_n(f) d\mu(f) \quad \text{is sufficiently small.}$$

We refer to the problem of finding a suitable probability measure as the *randomization problem.* It turns out that in analogy to (3.1) one can actually show that

$$b_i^{(2)}(M) \leq \mathrm{const}_n \, \mathbb{E}_{(\Omega,\mu)}(C_n) \quad \forall i \geq 0, \tag{3.2}$$

and that one can actually use the assumptions of Theorems 2.2 and 2.4 to construct a Γ-invariant probability measure μ such that $\mathbb{E}_{(\Omega,\mu)}(C_n)$ is smaller than $\mathrm{const}_n \mathrm{vol}(M)$ in the case of Theorem 2.2 and arbitrarily small in the case of Theorem 2.4, thus proving these theorems.

The construction of the latter will be explained in the toy case of $M = S^1$ in Subsection 3.4. A brief justification why (3.2) should hold follows next.

3.3. **L^2-Betti numbers and actions on probability spaces.** One would have to explain Damien Gaboriau's language of $\mathcal{R}$-simplicial complexes to give a proof of the estimate $b_i^{(2)}(M) \leq \mathbb{E}_{(\Omega,\mu)}(C_i)$. Instead, we want to at least point out that the L^2-Betti numbers of M can be computed by some sort of averaging over the probability space (Ω, μ). In Lück's algebraic definition averaging is reflected by interpreting $b_i^{(2)}(M)$ as the dimension of a certain induction of the homology of $\widetilde{M}$ with respect to a bigger von Neumann algebra, the so-called group measure construction of (Ω, μ) and Γ.

The *group measure space construction* $L^\infty(\Omega,\mu)\overline{\rtimes}\Gamma$ is defined as a completion of the algebraic crossed product $L^\infty(X) \rtimes \Gamma$ with respect to the trace

$$\mathrm{tr}\big(\sum f_\gamma \gamma\big) = \int_\Omega f_1(x) d\mu(x),$$

which is a sort of expected value. The group measure space construction contains the group von Neumann algebra $L(\Gamma)$ and $L^\infty(\Omega, \mu)$ as subalgebras. The crucial property is that

$$b_i^{(2)}(M) = \dim_{L^\infty(\Omega,\mu)\overline{\rtimes}\Gamma} H_i\big(L^\infty(\Omega,\mu)\overline{\rtimes}\Gamma \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M})\big) \tag{3.3}$$

For a proof of $b_i^{(2)}(M) \leq \mathrm{const}\, \mathbb{E}_{(\Omega,\mu)}(C_i)$ one would have to interpret the right hand side of (3.3) in Gaboriau's sense as L^2-Betti numbers of the $\mathcal{R}$-simplicial complex $\Omega \times \widetilde{M}$. For the better estimate (3.2) one needs a Poincare duality argument (see [4]).

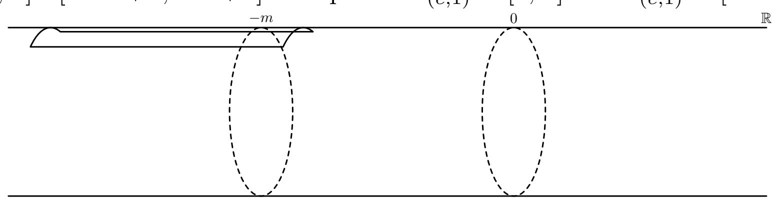
3.4. **The toy case $M = S^1$.** We want to execute the proof of Theorem 2.4, as presented in [5], for the example $M = S^1$. Of course, M itself is an amenable set, and we already know that its L^2-Betti numbers vanish. But we want to illustrate the construction of a $\mathbb{Z}$-invariant probability measure μ_ϵ on Ω such that the expected value $\mathbb{E}_{(\Omega,\mu_\epsilon)}(C_i)$ is smaller than a given $\epsilon > 0$.

Let $\Gamma = \mathbb{Z}$. For the index set I we take $I = \Gamma \times \{1, 2\}$. The measure μ on $\Omega = \operatorname{map}(\widetilde{M}, \Delta(I))$ will be obtained as the push-forward of the normalized Haar measure μ_{S^1} of S^1 under a certain Γ-equivariant map

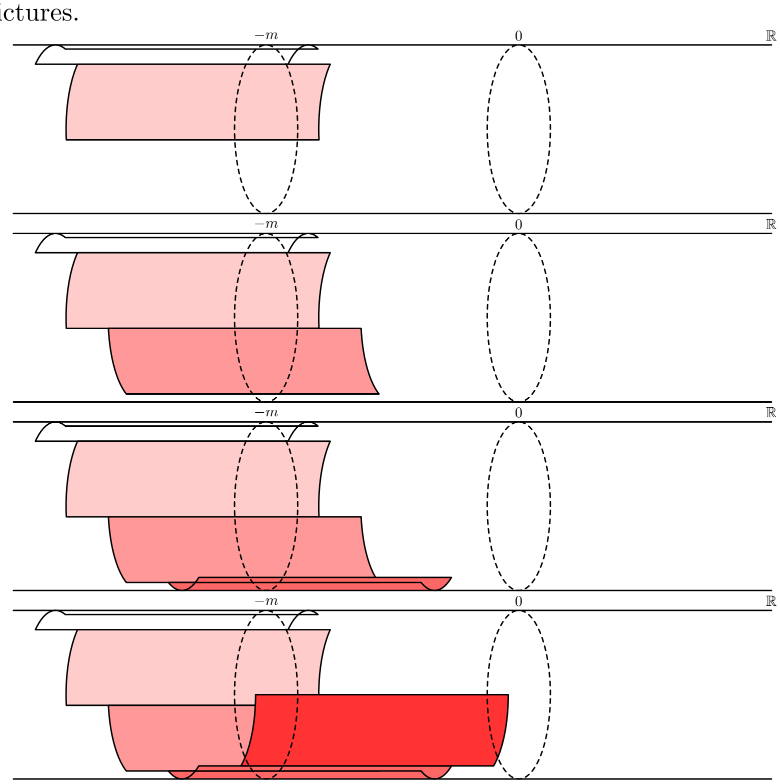
$$\phi_\epsilon : S^1 \to \Omega.$$

Let $m \in \mathbb{N}$ be larger than $2\epsilon^{-1}$. Let $\alpha \in [0, 1]$ be irrational with $0 < 1/m - \alpha < \frac{\epsilon}{2m}$. Equip $S^1 = \mathbb{R}/\mathbb{Z}$ with the ergodic rotation given by addition of α. Next we define an equivariant cover $\mathcal{U} = \{A_i \times U_i\}_{i \in I}$ of $S^1 \times \mathbb{R}$ such that $A_i \subset S^1$ are Borel sets and $U_i \subset \mathbb{R}$ are intervals of length m or 1. By definition, the map $\phi_\epsilon(z) : \mathbb{R} \to \Delta(I)$ is the nerve map associated to the cover $\{U_i;\ i \in I, z \in A_i\}$ for every $z \in S^1$.

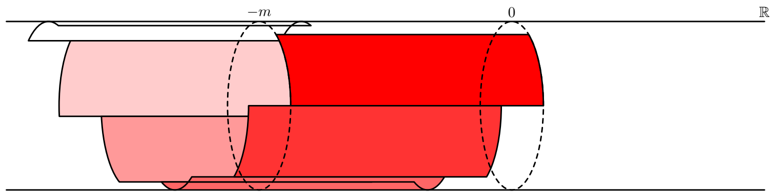
To describe $\mathcal{U}$, consider the following picture[2] of $S^1 \times \mathbb{R}$, where we see the tile $[0, \alpha] \times [-2m+1, -m+1]$ on top. Set $A_{(e,1)} = [0, \alpha]$ and $U_{(e,1)} = [-2m+1, -m+1]$.

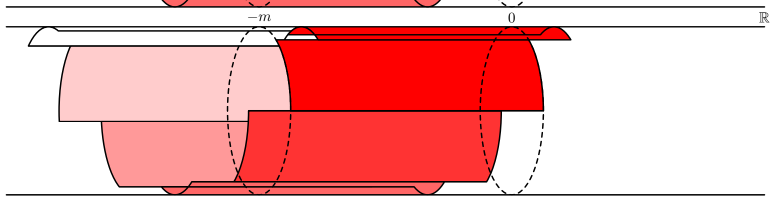


Next we consider the Γ-orbit $\{A_{(\gamma,1)} \times U_{(\gamma,1)}\}$ of the desribed tile in the following pictures.

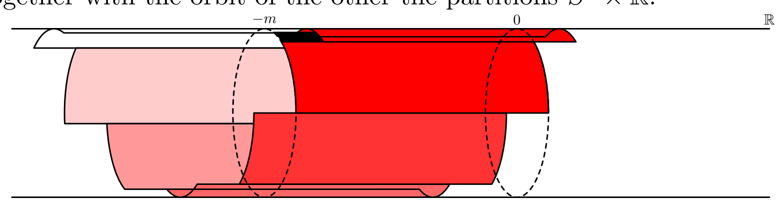


[2] I am grateful to Clara Löh for programming the pictures in Metafont.





We almost obtain a partition of the cylinder $S^1 \times \mathbb{R}$ but because of $m\alpha < 1$ the translates do not quite close up after m steps. We have to introduce another tile (black in the picture) $[1 - m\alpha] \times [-m, -m+1]$ whose Γ-orbit $\{A_{(\gamma,2)} \times U_{(\gamma,2)}\}$ together with the orbit of the other tile partitions $S^1 \times \mathbb{R}$.



Finally we make the tiles just a little bit longer in the $\mathbb{R}$-direction to obtain the desired cover. We leave it to reader to verify that

$$\mathbb{E}_{(\Omega,(\phi_\epsilon)_*\mu_{S^1})}(C_1) < 1 - m\alpha + \alpha < \epsilon.$$

3.5. **Final remarks.** In the actual proof of Theorems 2.2 and 2.4 one constructs suitable equivariant covers on the product of a Γ-probability space with $\widetilde{M}$, and then proceeds similarly as in Subsection 3.4 to obtain the desired probability measure on Ω. We want to mention the ingredients in the general case used to construct such covers.

In the case of Theorem 2.2 one can construct covers on $\widetilde{M}$ by balls of radius $0 < r < 1$ with multiplicity $< \mathrm{const}_n\, r^{-n}$ coming from maximal packings of concentric balls with smaller radii. This follows from the Bishop-Gromov inequality which provides packing inequalities in the presence of a lower Ricci curvature bound. In general, there is no way to obtain equivariant such covers. However, a suitable randomization in the sense of Subsection 3.2 of the problem of the existence of equivariant covers with small multiplicity can be solved, which leads to a proof of Theorem 2.2.

In the case of Theorem 2.4 one applies the generalized Rokhlin lemma from ergodic theory to construct covers similar to the one in the toy example over every of the amenable subsets and combines them to a cover on the product of a Γ-probablity space and $\widetilde{M}$.

References

[1] Michael Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. (1982), no. 56, 5–99 (1983).

[2] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser, 1999.

[3] Wolfgang Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 44, Springer-Verlag, 2002.

[4] Marco Schmidt, *L^2-Betti numbers of $\mathscr{R}$-spaces and the Integral Foliated Simplicial Volume*, Universität Münster, 2005. doctoral thesis.

[5] Roman Sauer, *Amenable Covers, Volume and L^2-Betti Numbers of Aspherical Manifolds*, available at `math.AT/0605627`.