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5.3 Euler characteristic (Beginning of "Algebraic topology 2")

A non-zero ring R is a domain if $ab=0$ implies $a=0$ or $b=0$ for $a,b \in R$, i.e. R has no zero-divisors.

A commutative domain is called an integral domain.

A principal ideal domain (PID) is an integral domain in which every ideal is generated by a single element.

Examples: fields, $\mathbb{Z}$, $k[X]$ for a field k are PIDs.

The quotient field $Q(R)$ of an integral domain R is the set

$$\left\{ \frac{a}{b} \mid a \in R,\ b \in R\setminus\{0\} \right\}$$

of equivalence classes $\frac{a}{b}$ of pairs with respect to $\frac{a}{b} = \frac{a'}{b'} \iff ab' = a'b$ with the usual rules of addition and multiplication for fractions.

These turn $Q(R)$ into a field in which R is embedded as a ring $R \hookrightarrow Q(R)$.

The structure theorem on finitely generated modules over a PID says that a finitely generated R-module M, R PID, splits as

$$M \cong R^n \oplus \operatorname{tors}(M),$$

where $\operatorname{tors}(M) = \{ m \in M \mid \exists r \in R\setminus\{0\} : r\cdot m = 0 \}$.

The number n is uniquely determined, and it is called the rank $\operatorname{rk}_R(M) \in \mathbb{N}_0$ of M.

Later we'll see that $\operatorname{rk}_R(M) = \dim_{Q(R)}(Q(R) \otimes_R M)$.

↺

If $0 \to M_0 \to M_1 \to M_2 \to 0$ is a short exact sequence of R-modules, then

(additivity) $\quad \mathrm{rk}_R(M_1) = \mathrm{rk}_R(M_0) + \mathrm{rk}_R(M_2)$.

This will follow from two facts:

- $Q(R) \otimes_R -$ is exact, i.e. preserves exactness of sequences
- vector space dimension is additive

Def.: let X be a finite CW-complex.
The Euler characteristic $\chi(X) \in \mathbb{Z}$ is defined as

$$\chi(X) = \sum_{p \geq 0} (-1)^p (\# \text{ p-cells in } X)$$

Def.: The p-th Betti number of a space X with coefficients in a PID R is $b_p(X;R) = \mathrm{rk}_R(H_p(X;R)) \in \mathbb{N}_0 \cup \{\infty\}$.
For $R = \mathbb{Z}$, we write $b_p(X) = b_p(X;\mathbb{Z})$.

Remark: If X is a CW-complex with finitely many p-cells, then $b_p(X;R) < \infty$. Reason:
$H_p(X;R) = H_p(C_*^{CW}(X;R))$ is a subquotient of
$C_p^{CW}(X;R) \cong R^{\# \text{p-cells}}$, hence has finite R-rank.

↗

Theorem (Euler - Poincaré - formula):

Let X be a finite CW complex. Let R be a PID.

Then:
$$\chi(X) = \sum_{p \geq 0} (-1)^p \, b_p(X; R)$$

Proof: Let $C_* := C_*^{CW}(X; R)$. Note that

$$\# \, p\text{-cells in } X = \operatorname{rk}_R(C_p) .$$

To show:
$$\sum_{p\geq 0} (-1)^p \operatorname{rk}_R(C_p) = \sum_{p \geq 0} (-1)^p \operatorname{rk}_R(H_p(C_*)) \qquad (*)$$

We have two short exact sequences of R-modules:

$$\begin{tikzcd} 0 \arrow[r] & B_p \arrow[r] & Z_p \arrow[r] & H_p(C_*) \arrow[r] & 0 \end{tikzcd}$$

$Z_p \subset C_p$ cycles
$B_p \subset C_p$ boundaries

$$\begin{tikzcd} 0 \arrow[r] & Z_p \arrow[r] & C_p \arrow[r, "\partial_p"] & B_{p-1} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{aligned}
\sum_{p\geq 0} (-1)^p \operatorname{rk}_R(C_p) &= \sum_{p \geq 0} (-1)^p \left(\operatorname{rk}_R(Z_p) + \operatorname{rk}_R(B_{p-1}) \right) \\
&= \sum_{p \geq 0} (-1)^p \left(\operatorname{rk}_R(H_p(C_*)) + \operatorname{rk}_R(B_p) + \operatorname{rk}_R(B_{p-1}) \right) \\
&= \sum_{p\geq 0} (-1)^p \operatorname{rk}_R(H_p(C_*)) . \qquad \Longrightarrow (*)
\end{aligned}$$

▨

Cor.: The Euler characteristic is a homotopy invariant.

Rem.:
$$\sum_{p=0}^{n} (-1)^p \binom{n+1}{p+1} = 1 .$$

$$\chi(\Delta^n) = \sum_{p \geq 0} (-1)^p \, b_p(\Delta^n) = 1$$
$$\parallel$$
$$\sum_{p=0}^{n} (-1)^p (\# \, p\text{-simplices}) = \sum_{p=0}^{n} (-1)^p \binom{n+1}{p+1} .$$

↺

Thm.: Let X, Y be two finite CW complexes.

Then $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Proof: Let $a_p(X) := \#\ p\text{-cells in } X$.

We have $a_p(X \times Y) = \sum_{i=0}^{p} a_i(X) \cdot a_{p-i}(Y)$. (see section 5.1)

$$\chi(X \times Y) = \sum_{p \geq 0} (-1)^p a_p(X \times Y) = \sum_{p \geq 0} \sum_{i=0}^{p} a_i(X) \cdot a_{p-i}(Y)\ (-1)^p$$

$$= \sum_{p \geq 0} \sum_{i=0}^{p} a_i(X)(-1)^i\ a_{p-i}(Y)(-1)^{p-i} .$$ ▨

Theorem: Let A, X, Y be finite CW complexes.

Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

be a pushout

s.th. i closed embedding and $i(A) \subset X$ neighbourhood deformation retract.

(We do <u>not</u> assume that i, f respect the CW structures.)

Then $\chi(Z) = \chi(X) + \chi(Y) - \chi(A)$. (Additivity)

Proof: Consider the MV sequence

$$\ldots \to \overset{C_5}{H_1(A)} \to \overset{C_4}{H_1(X) \oplus H_1(Y)} \to \overset{C_3}{H_1(Z)} \to$$

$$\to \underset{C_2}{H_0(A)} \to \underset{C_1}{H_0(X) \oplus H_0(Y)} \to \underset{C_0}{H_0(Z)} \to 0$$

and regard it as an exact chain complex C_*. We have $H_p(C_*) = 0$ for all p .

↗

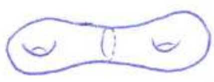
By (*), $0 = \sum_{p\geq 0} (-1)^p \mathrm{rk}_{\mathbb{Z}} H_p(C_*) = \sum_{p\geq 0} (-1)^p \mathrm{rk}_{\mathbb{Z}}(C_p)$

$= \mathrm{rk}_{\mathbb{Z}}(C_0) - \mathrm{rk}_{\mathbb{Z}}(C_1) + \mathrm{rk}_{\mathbb{Z}}(C_2) + \dots$

$= \mathrm{rk}_{\mathbb{Z}}(H_0(Z)) - (\mathrm{rk}_{\mathbb{Z}}(H_0(X)) + \mathrm{rk}_{\mathbb{Z}}(H_0(Y))) + \mathrm{rk}_{\mathbb{Z}} H_0(A) + \dots$

$= \chi(Z) - (\chi(X) + \chi(Y)) + \chi(A)$. $\square$

Examples :

- $\chi(S^1) = 0$
- $\chi(S^1 \times S^1) = 0$
- $\chi(S^2) = 2$
- $\chi\Big($  $\Big) = 2 \cdot \chi\big($  $\big) - \chi(S^1)$

  $= 2 \cdot \chi(S^1 \vee S^1)$

  $= 2\,(2\chi(S^1) - \chi(\cdot)) = -2$
- $\chi\Big($  $\Big) = 2 - 2g$

  g holes
- $$\chi(M) = \frac{1}{2\pi} \int_M K \, d\,\mathrm{vol}_{(M,g)}$$ (Gauss-Bonnet formula)

  for a surface M with Riem. metric g.

Remark: (Hopf-Singer conjecture)

Let M be an aspherical closed manifold (i.e. universal covering $\widetilde{M}$ is contractible) - e.g. if M admits a Riem. metric of non-positive curvature - then :

↺

$$\begin{cases} \chi(M) = 0, & \text{if } \dim M \text{ odd}, \\ (-1)^d \chi(M) \geq 0, & \text{if } \dim M = 2d \text{ even}. \end{cases}$$

20.10.

6. Homological algebra

The main intuition in homological algebra ("algebra of manipulating chain complexes") comes from topology.

6.1 Projective modules

Let R be a (not necessarily commutative) ring.
Unless stated otherwise, we consider left R-modules.

Def.: An R-module P is called projective if it has the following property:

$$\begin{tikzcd} M \arrow[r, "p"] & N \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[ul, dashed, "\tilde{f}"] & \end{tikzcd}$$

For every epimorphism $M \xrightarrow{p} N$ and a homomorphism $P \xrightarrow{f} N$ there is a homomorphism $P \xrightarrow{\tilde{f}} M$ s.th. $p \circ \tilde{f} = f$.

Theorem: The following are equivalent:

1) P is projective.

2) Every short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & M \arrow[r, "p"] & P \arrow[r] \arrow[l, bend left, dashed, "s"] & 0 \end{tikzcd}$$

splits, i.e. there is a homomorphism $s: P \to M$ s.th. $p \circ s = id_P$. Moreover, $N \oplus P \xrightarrow{i \oplus s} M$ is an isomorphism.

3) P is a direct summand in a free R-module F.

If P, in addition, is finitely generated, then F can be taken to be finitely generated.

Proof: 1) $\Rightarrow$ 2) : We solve the lifting problem for id_P :

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "i"] & M \arrow[r, "p"] & P \arrow[r] & 0 \\
 & & & P \arrow[u, "id_P"'] \arrow[ul, dashed, "s"] &
\end{tikzcd}$$

Injectivity of $i \oplus s$:

Assume $(i\oplus s)(n,x) = i(n) + s(x) = 0$

$\Rightarrow \quad 0 = p(i(n)+s(x)) = p(i(n)) + p(s(x)) = 0 + x = x$

i injective $\Rightarrow$ $n=0$.

Surjectivity of $i\oplus s$:

Let $x \in M$. Consider $x - s\circ p(x)$.

We have $p(x - s\circ p(x)) = p(x) - p(x) = 0$.

$\Rightarrow$ $x - s\circ p(x) \in \ker(p) = \mathrm{im}(i)$.

2) $\Rightarrow$ 3): Choose an epimorphism $F \overset{q}{\twoheadrightarrow} P$ for a free module F.

Now $0 \to \ker(p) \to F \twoheadrightarrow P \to 0$ splits,

i.e. $P \oplus \ker(p) \cong F$.

3) $\Rightarrow$ 1):

$$\begin{tikzcd}
M \arrow[r, two heads] & N & \\
 & P \arrow[u, "f"] \arrow[ul, dashed, "\tilde{f}"] & P\oplus P' \cong F \arrow[l, "proj"'] \arrow[ul, bend right, "g"'] \arrow[ull, bend left=40, "\tilde{g}"]
\end{tikzcd}$$

The restriction $\tilde{f}$ of $\tilde{g}$ to $P \oplus 0$ solves the extension problem. ↺

Examples:

1) $R = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z}$ is not projective.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

does not split.

2) $R = \mathbb{F}_2 \times \mathbb{F}_2$

$P = \mathbb{F}_2 \times \{0\}$ is projective since $P \oplus (\{0\} \times \mathbb{F}_2) \cong R$

but not free since P has just 2 elements (not a power of 4).

3) $R = C(S^2) = \{ f: S^2 \to R \mid f \text{ cont.} \}$

Over R there is an example of a finitely generated projective module P s.th. $P \oplus R \cong \mathbb{R}^3$ ("stably free") but P is not free.

(Top. K-Theory of X $\longleftrightarrow$ study of proj. modules over $C(X)$)

TS^2 "tangent bundle" (example of vector bundle)

$$\begin{tikzcd} TS^2 = \bigcup_{p\in S^2} T_pS^2 = \{(s,v) \mid s \perp v\} \subset \mathbb{R}^3 \times \mathbb{R}^3 \arrow[r, "pr"] & S^2 \end{tikzcd}$$

$P := \Gamma(TS^2) = \{ s: S^2 \to TS^2 \mid s \text{ continuous}, \; pr \circ s = id_{S^2} \}$

Claim: P is projective.

$$P \oplus C(S^2) \longrightarrow C(S^2)^3 \cong C(S^2, \mathbb{R}^3)$$
$$(s, f) \longmapsto (x \longmapsto \underbrace{s(x) + f(x)\cdot x}_{\in \mathbb{R}^3})$$

↗

But P is not free! [Rosenberg: K-theory]

(Underlying reason: Every vector field over S^2 has a zero.)

6.2 The fundamental theorem of homological algebra

Def.: Let M be an R-module. A resolution of M is an exact chain complex

$$\dots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \xrightarrow{\varepsilon} M \longrightarrow 0 .$$

(Equivalently, an R-chain complex M_* with an isomorphism $H_0(M_*) \xrightarrow{\cong} M$ and $H_i(M_*) = 0$ for $i > 0$.)

A projective resolution P_* of M is a resolution s.th. $P_0, P_1, \dots$ are projective.

Theorem:

1) Every R-module has a projective resolution.
2) Let P_* be a chain complex of projective modules. Let Q_* be a chain complex s.th. $H_i(Q_*) = 0$ for $i > 0$. Let $[P_*, Q_*]$ be the abelian group of chain homotopy classes of chain maps $P_* \to Q_*$.

   Then
   $$[P_*, Q_*] \xrightarrow{\cong} \hom_R(H_0(P_*), H_0(Q_*))$$
   $$[f_*] \longmapsto \big(H_0(f_*): H_0(P_*) \to H_0(Q_*)\big)$$
   is an isomorphism.

Cor.: Two projective resolutions P_* and Q_* of the same R-module are chain homotopy equivalent: $P_* \simeq Q_*$.

↺

Proof:

$$\begin{tikzcd}
 & & & H_0(Q_*) \arrow[d, phantom, "\cong"] & \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0 \\
\cdots \arrow[r] \arrow[u, "f_2"] & P_1 \arrow[r] \arrow[u, "f_1"] & P_0 \arrow[r] \arrow[u, "f_0"] & M \arrow[r] \arrow[u, "id_M = H_0(f_*)"'] & 0 \\
 & & & H_0(P_*) \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

By surjectivity, we obtain f_* as above.

Reversing the order, we obtain a chain map

$g_* : Q_* \longrightarrow P_*$.

$g_* \circ f_*$ and $f_* \circ g_*$ induces id_M on H_0.

By injectivity, we get $\begin{cases} g_* \circ f_* \simeq id_{P_*} \\ f_* \circ g_* \simeq id_{Q_*} \end{cases}$

Proof of the theorem:

1) Let M be an R-module.

- Choose e.g. a free R-module P_0 surjecting onto M:

$$P_0 \twoheadrightarrow M$$

-
$$\begin{tikzcd}
 & \ker(\varepsilon) \arrow[dr, hook] & & & \\
P_1 \arrow[ur, two heads] \arrow[rr] & & P_0 \arrow[r, two heads, "\varepsilon"] & M \arrow[r] & 0
\end{tikzcd}$$

-
$$\begin{tikzcd}
 & \ker(d) \arrow[dr, hook] & & \\
P_n \arrow[ur, two heads] \arrow[rr] & & P_{n-1} \arrow[r, "d"] & P_{n-2}
\end{tikzcd}$$

But P is not free! [Rosenberg : K-theory]

(Underlying reason: Every vector field over S^2 has a zero.)

6.2 The fundamental theorem of homological algebra

Def.: Let M be an R-module. A resolution of M is an exact chain complex

$$\cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \xrightarrow{\varepsilon} M \longrightarrow 0 .$$

(Equivalently, an R-chain complex M_* with an isomorphism $H_0(M_*) \xrightarrow{\cong} M$ and $H_i(M_*) = 0$ for $i > 0$.)

A projective resolution P_* of M is a resolution s.th. $P_0, P_1, \ldots$ are projective.

Theorem:

1) Every R-module has a projective resolution.
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Then
$$[P_*, Q_*] \xrightarrow{\cong} \hom_R(H_0(P_*), H_0(Q_*))$$
$$[f_*] \longmapsto \left(H_0(f_*) : H_0(P_*) \to H_0(Q_*)\right)$$
is an isomorphism.

Cor.: Two projective resolutions P_* and Q_* of the same R-modules are chain homotopy equivalent: $P_* \simeq Q_*$.

↺

Proof:

$$\begin{tikzcd}
 & & & H_0(Q_*) \arrow[d, phantom, "\cong"] & \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0 \\
\cdots \arrow[r] \arrow[u, "f_2"] & P_1 \arrow[r] \arrow[u, "f_1"] & P_0 \arrow[r] \arrow[u, "f_0"] & M \arrow[r] \arrow[u, "id_M = H_0(f_*)"'] & 0 \\
 & & & H_0(P_*) \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

By surjectivity, we obtain f_* as above.

Reversing the order, we obtain a chain map

$g_* : Q_* \longrightarrow P_*$.

$g_* \circ f_*$ and $f_* \circ g_*$ induce id_M on H_0.

By injectivity, we get $\begin{cases} g_* \circ f_* \simeq id_{P_*} \\ f_* \circ g_* \simeq id_{Q_*} \end{cases}$

Proof of the theorem:

1) Let M be an R-module.

- Choose e.g. a free R-module P_0 surjecting onto M:

$$P_0 \twoheadrightarrow M$$

-

$$\begin{tikzcd}
 & \ker(\varepsilon) \arrow[dr, hook] & & & \\
P_1 \arrow[ur, two heads] \arrow[rr] & & P_0 \arrow[r, two heads, "\varepsilon"] & M \arrow[r] & 0
\end{tikzcd}$$

-

$$\begin{tikzcd}
 & \ker(d) \arrow[dr, hook] & & \\
P_n \arrow[ur, two heads] \arrow[rr] & & P_{n-1} \arrow[r, "d"] & P_{n-2}
\end{tikzcd}$$

This is a projective resolution for M.

2) To show: $[P_*, Q_*] \xrightarrow{\cong} hom_R(H_0(P_*), H_0(Q_*))$

$$[f_*] \longmapsto H_0(f_*)$$

↗

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Surjectivity:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "f_2"] & P_1 \arrow[r, "p_1"] \arrow[d, dashed, "f_1"] & P_0 \arrow[r, two heads, "\varepsilon"] \arrow[d, dashed, "f_0"] & H_0(P_*) \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "q_2"] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, two heads, "\eta"] & H_0(Q_*) \arrow[r] & 0
\end{tikzcd}$$

Step 0: Take f_0 to be a solution of this lifting problem:

$$\begin{tikzcd}
P_0 \arrow[d, dashed] \arrow[dr, bend left, "\varphi \circ \varepsilon"] & & \\
Q_0 \arrow[r, two heads, "\eta"] & H_0(Q_*) \arrow[r] & 0
\end{tikzcd}$$

(P_0 projective)

Step 1: Use

$$\begin{tikzcd}
P_1 \arrow[d, dashed, "f_1"] \arrow[dr, bend left, "f_0 \circ p_1"] & \\
Q_1 \arrow[r, two heads] & Ker(\eta)
\end{tikzcd}$$

for f_1. (P_1 projective)

Proceed inductively:

$$\begin{tikzcd}
P_{n+1} \arrow[d, dashed, "f_{n+1}"] \arrow[dr, bend left, "f_n \circ p_{n+1}"] & \\
Q_{n+1} \arrow[r, two heads] & ker(q_n) = im(q_{n+1})
\end{tikzcd}$$

(P_{n+1} projective)

Note $im(f_n \circ p_{n+1}) \subset ker(q_n)$ since

$$q_n \circ f_n \circ p_{n+1} = f_{n-1} \circ p_n \circ p_{n+1} = 0 .$$

Injectivity: Given two chain maps

$$f_*, g_* : P_* \to Q_*$$

with $H_0(f_*) = H_0(g_*) =: \varphi$, we have to show the existence of a chain homotopy $h_* : P_* \to Q_{*+1}$, $h_* : f_* \simeq g_*$, i.e. ↺

$$q_{n+1} \circ h_n + h_{n-1} \circ p_n = f_n - g_n$$

$$\begin{tikzcd}
\dots \arrow[r] & P_1 \arrow[r] \arrow[dl, dashed, "h_1"'] \arrow[d, "{f_1,g_1}"] & P_0 \arrow[r, "\varepsilon"] \arrow[dl, dashed, "h_0"'] \arrow[d, "{f_0,g_0}"] & H_0(P_*) \arrow[r] \arrow[d, "\varphi"] & 0 \\
Q_2 \arrow[r] & Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, "\eta"] & H_0(Q_*) \arrow[r] & 0
\end{tikzcd}$$

Step 0:

$$\begin{tikzcd}
 & P_0 \arrow[dl, dashed, "h_0"'] \arrow[d] \\
Q_1 \arrow[r, two heads, "q_1"'] & \ker(\eta) = \operatorname{im}(q_1)
\end{tikzcd}$$

Take h_0 to be the solution of this lifting problem. (Q_1 projective)

Step 1:

$$q_2 \circ h_1 = f_1 - g_1 - h_0 \circ p_1$$

Therefore, h_1 is the solution of

$$\begin{tikzcd}
 & P_1 \arrow[dl, dashed, "h_1"'] \arrow[d, "f_1 - g_1 - h_0 \circ p_1"] \\
Q_2 \arrow[r, two heads, "q_2"'] & \ker(q_1) = \operatorname{im}(q_2)
\end{tikzcd}$$

.

Note for that:

$$\begin{aligned} q_1(f_1 - g_1 - h_0 \circ p_1) &= f_0 \circ p_1 - g_0 \circ p_1 - q_1 \circ h_0 \circ p_1 \\ &= f_0 \circ p_1 - g_0 \circ p_1 - (f_0 - g_0) \circ p_1 \\ &= 0 . \end{aligned}$$

Now proceed inductively. ☒

↗

6.3 Exactness of functors

Def.: A (covariant) functor F from the category of (left or right) R-modules (denoted by R-mod resp. Mod-R) to abelian groups is called

1) left exact: if for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

the sequence

$$0 \to F(M_1) \to F(M_2) \to F(M_3)$$

is exact.

2) right exact, if ——— " ——— , the sequence

$$F(M_1) \to F(M_2) \to F(M_3) \to 0$$

is exact.

3) exact, if F is left and right exact.

Further, a contravariant functor F from R-Mod to Ab is (left / right) exact if the associated functor $R\text{-Mod}^{op} \to Ab$ or $\text{Mod-}R^{op} \to Ab$ is (left / right) exact. E.g.:

$$F \text{ left exact } \Leftrightarrow \left(\begin{array}{l} 0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow 0 \text{ exact} \\ \Rightarrow 0 \to F(M_1) \to F(M_2) \to F(M_3) \text{ exact} \end{array} \right)$$

↺

Proposition: Let M be an (left) R-module. The functor

$$\begin{aligned} \hom_R(M,-) : \quad & R\text{-}Mod \longrightarrow Ab \\ & N \longmapsto \hom_R(M,N) \end{aligned}$$

is left exact.

Proof: Consider an exact sequence

$$0 \to N_1 \xhookrightarrow{j} N_2 \xrightarrow{p} N_3 \longrightarrow 0$$

of R-modules. To show:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \hom_R(M,N_1) & \xrightarrow{j_*} & \hom_R(M,N_2) & \xrightarrow{p_*} & \hom_R(M,N_3) & \longrightarrow 0 \\ & \varphi & \longmapsto & j\circ\varphi & & & \\ & & & \Psi & \longmapsto & p\circ\Psi & \end{array}$$

is exact.

- j_* injective: $0 = j_*(\varphi) \Rightarrow j(\varphi(x)) = 0 \quad \forall x\in M$
  $\overset{j \text{ inj.}}{\Rightarrow} \varphi(x) = 0 \quad \forall x\in M$
  $\Rightarrow \varphi = 0$
- $\mathrm{im}(j_*) \subset \ker(p_*)$: ✓
- $\ker(p_*) \subset \mathrm{im}(j_*)$: Let $\Psi \in \ker(p_*)$, so $p\circ\Psi = 0$.

  So for every $x\in M$, $\Psi(x) \in \ker(p) = \mathrm{im}(j)$.

  Hence $\Psi = j_*(\varphi)$, where $\varphi = (N_1 \xrightarrow{j} \mathrm{im}(j))^{-1}\circ\Psi$.

$\boxtimes$

Proposition: Let P be a projective R-module. Then $\hom_R(P,-)$ is exact.

Proof: Enough to show: $\hom_R(P,-)$ preserves surjectivity.

This is just a reformulation of the definition of a projective module. ↗

Actually, it holds: " P projective $\iff$ $\hom_R(P,-)$ exact ".

__Def.:__ An R-module I is __injective__ if $\hom_R(-,I)$ is exact.

__Example:__ 1) $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module since

$$\begin{tikzcd}
& \mathbb{Z} & \\
0 \arrow[r] & \mathbb{Z} \arrow[u, "id"] \arrow[r, "\cdot 2"'] & \mathbb{Z} \arrow[ul, dotted, bend right]
\end{tikzcd}$$

cannot be solved.

2) $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

[see Weibel; Introduction to homological algebra, p.40]

6.4 __Tensor products__

__Def.:__ Let M be a right R-module, N be a left R-module and X be an abelian group.

A __bilinear homomorphism__ $f: M \times N \to R$ is a map s.th.

- for fixed $n \in N$, $f(-,n): M \to X$ is a group homomorphism
- for fixed $m \in M$, $f(m,-): N \to X$ — " —
- for $r \in R$, $m \in M$, $n \in N$, $f(m \cdot r, n) = f(m, r \cdot n)$.

__Def.:__ Let M, N as above.

The tensor product $M \otimes_R N$ of M and N is an abelian group together with a bilinear homomorphism $b: M \times N \to M \otimes_R N$ such that the following universal property holds:

> For any bilinear homomorphism $f: M \times N \to X$ there is a unique homom. of abelian groups $M \otimes_R N \to X$ s.th. the diagram commutes:
>
> $$\begin{tikzcd}
M \times N \arrow[r, "f"] \arrow[d, "b"'] & X \\
M \otimes_R N \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

5

Proof of existence :

Take the free abelian group $F(M \times N)$ over the set $M \times N$.

Let $\mathcal{U} \subset F(M \times N)$ be the subgroup generated by elements

$$\begin{cases} (m+m', n) - (m,n) - (m',n) \\ (m, n+n') - (m,n) - (m,n') \\ (m \cdot r, n) - (m, r \cdot n) \end{cases}$$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$.

Set $M \underset{R}{\otimes} N := F(M \times N) / \mathcal{U}$ and $b((m,n)) = [(m,n)]$ for $m \in M, n \in N$.

Then $M \underset{R}{\otimes} N$ and b have the desired properties. $\boxtimes$

Notation : An element $b((m,n))$ is called an elementary tensor and denoted by $m \otimes n$.

Def.: Let R and S be two rings. An R-S-bimodule is an abelian group M such that M is a left R-module and a right-S-module (with respect to this abelian group structure) s.th. $r(m \cdot s) = (rm) \cdot s$ for all $r \in R$, $s \in S$, $m \in M$.

Examples: 1) $\mathbb{Q}^n$ is a $\mathbb{Q}$-$M_n(\mathbb{Q})$-bimodule.

2) For commutative R, every left R-module becomes a right R-module via $m \cdot r := rm$ and an R-R-bimodule.

3) If $R \subset S$ is a subring, then S is an R-R-bimodule, R-S-bimodule . .

↗

Rem.: 1) If M is an S-R-bimodule and N is a left R-module, then $M \underset{R}{\otimes} N$ becomes a left S-module via $s(m \otimes n) = sm \otimes n$. Further, if L is a right R-module, then $\hom_R(M, L)$ becomes a right S-module via $(f \cdot s)(m) = s f(m)$.

27.10.

2) If R is a commutative ring, then $M \underset{R}{\otimes} N$ is an R-module. Moreover, it satisfies then the universal property:

$$\begin{tikzcd} M \times N \arrow[r, "b"] \arrow[dr, "f"'] & M \underset{R}{\otimes} N \arrow[d, dashed, "\exists! \text{ R-linear}"] \\ & X \end{tikzcd}$$

for every bilinear f with $f(m \cdot r, n) = f(m, r \cdot n) = r \cdot f(m,n)$.

3) From the universal property we also obtain that

$M \underset{R}{\otimes} - : R\text{-}Mod \longrightarrow Ab$

$- \underset{R}{\otimes} N : Mod\text{-}R \longrightarrow Ab$

are functors, since $g: W \to W'$ R-homomorphism yields

$$\begin{tikzcd} M \times W \arrow[r] \arrow[d, "id \times g"'] & M \underset{R}{\otimes} W \arrow[d, "id \otimes g"] \\ M \times W' \arrow[r] & M \underset{R}{\otimes} W' \end{tikzcd}$$

On elementary tensors: $(id \otimes g)(x \otimes y) = x \otimes g(y)$.

Similarly, for an S-R-bimodule M we obtain a functor

$M \underset{R}{\otimes} - : R\text{-}Mod \to S\text{-}Mod$ etc.

↻

Also from universal property we get :

<u>Proposition</u>: Let B be a left R-module. Then the functor

$$- \underset{R}{\otimes} B : \quad \mathrm{Mod}\text{-}R \longrightarrow Ab$$

is left adjoint to $\hom_{\mathbb{Z}}(B, -) : Ab \to \mathrm{Mod}\text{-}R$.

If, in addition, B is a R-S-bimodule, then

$$- \underset{R}{\otimes} B : \quad \mathrm{Mod}\text{-}R \longrightarrow \mathrm{Mod}\text{-}S$$

is left adjoint to $\hom_S(B, -) : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$.

Explicitly, we have natural isomorphisms

$$\hom_{\mathbb{Z}}(A \underset{R}{\otimes} B, C) \cong \hom_R(A, \hom_{\mathbb{Z}}(B, C))$$

and

$$\hom_S(A \underset{R}{\otimes} B, C) \cong \hom_R(A, \hom_S(B, C)) .$$

<u>Proposition</u>: Let B be a left R-module.

The functor $- \underset{R}{\otimes} B$ is right exact.

<u>Proof</u>: Assume $0 \to A_1 \xrightarrow{j} A_2 \xrightarrow{p} A_3 \to 0$ is exact.

Denote it by $\underline{A}$, and

$$A_1 \underset{R}{\otimes} B \to A_2 \underset{R}{\otimes} B \to A_3 \underset{R}{\otimes} B \to 0$$

by $\underline{A} \otimes B$. For every $\mathbb{Z}$-module C, the sequence

$$\hom_R(\underline{A} \otimes B, C) \cong \hom_R(\underline{A}, \hom_{\mathbb{Z}}(B, C))$$

is exact since $\hom_R(-, \hom_{\mathbb{Z}}(B, C))$ is left exact.

This implies that $\underline{A} \otimes B$ is exact as follows :

↗

Let $C = {A_2 \otimes_R B} \big/ {\mathrm{im}(j \otimes \mathrm{id})}$.

Consider the projection $(A_2 \otimes_R B \xrightarrow{pr} C) \in \hom_R(A_2 \otimes_R B, C)$;

it is mapped to zero in $\hom_R(\underline{A} \otimes B, C)$.

By exactness, there is $f \in \hom_R(A_3 \otimes_R B, C)$ with

$$\left(A_2 \otimes_R B \xrightarrow{p \otimes id} A_3 \otimes_R B \xrightarrow{f} C \right) = pr .$$

Hence, $\ker(p \otimes id) \subset \ker(pr) = \mathrm{im}(j \otimes id)$ and so $\ker(p \otimes id) = \mathrm{im}(j \otimes id$

This is left to the reader. □

Prop.: (Basic properties)

1) $R \otimes_R N \longrightarrow N$ isomorphism of left R-modules
$r \otimes n \longmapsto rn$

2) $M \otimes_R R \longrightarrow M$ isomorphism of right R-modules
$m \otimes r \longmapsto mr$

3) $$M \otimes_R \left(\bigoplus_{i \in I} N_i \right) \xrightarrow{\cong} \bigoplus_{i \in I} (M \otimes_R N_i)$$
$$m \otimes (n_i)_{i \in I} \longmapsto (m \otimes n_i)_{i \in I}$$

4) $$\left(\bigoplus_{i \in I} M_i \right) \otimes_R N \xrightarrow{\cong} \bigoplus_{i \in I} (M_i \otimes_R N)$$

Examples: 1) In $M \otimes_R N$ we have $0 \otimes n = m \otimes 0 = 0$
since $0 \otimes n = 0 \cdot 0_R \otimes n = 0 \otimes 0_R \cdot n = 0$.

2) Let A be a finite abelian group. Then $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$:
There is $n \in \mathbb{Z} \setminus \{0\}$ with $n \cdot a = 0 \quad \forall a \in A$.
Hence $\frac{x}{y} \otimes a = \frac{x}{yn} \cdot n \otimes a = \frac{x}{yn} \otimes na = 0$.

↺

3) $\mathbb{Z}/_3 \otimes_{\mathbb{Z}} \mathbb{Z}/_5 = 0$ (since 3 is invertible in $\mathbb{Z}/_5$)

4) let $d = \gcd(a,b)$. Then $\mathbb{Z}/_a \otimes_{\mathbb{Z}} \mathbb{Z}/_b \cong \mathbb{Z}/_d$.

Choose projective resolution of $\mathbb{Z}/_a$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \longrightarrow \mathbb{Z}/_a \longrightarrow 0$$

Take $- \otimes_{\mathbb{Z}} \mathbb{Z}/_b$. Then

$$\begin{tikzcd} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/_b \arrow[r] \arrow[d, phantom, "\wr"] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/_b \arrow[r] \arrow[d, phantom, "\wr"] & \mathbb{Z}/_a \otimes_{\mathbb{Z}} \mathbb{Z}/_b \arrow[r] & 0 \\ \mathbb{Z}/_b \arrow[r, "\cdot a"] & \mathbb{Z}/_b & & \end{tikzcd}$$

is exact.

So $\mathbb{Z}/_a \otimes_{\mathbb{Z}} \mathbb{Z}/_b \cong \operatorname{coker}\left(\mathbb{Z}/_b \xrightarrow{\cdot a} \mathbb{Z}/_b \right)$.

Also
$$\begin{tikzcd} \mathbb{Z}/_b \arrow[r, "\cdot a"] & \mathbb{Z}/_b \arrow[r] & \mathbb{Z}/_d \arrow[r] & 0 \\ & {[x]} \arrow[r, mapsto] & 0 & \end{tikzcd}$$

$\Rightarrow \quad x = d \cdot e = (an + bm) \cdot e = a \cdot ne$

$\Rightarrow \quad [x] = a\,[ne]$

Hence $\mathbb{Z}/_d \cong \operatorname{coker}\left(\mathbb{Z}/_b \xrightarrow{\cdot a} \mathbb{Z}/_b \right) \cong \mathbb{Z}/_a \otimes_{\mathbb{Z}} \mathbb{Z}/_b$.

5) $\left(\mathbb{Z}^2 \oplus \mathbb{Z}/_{17} \oplus \mathbb{Z}/_2\right) \otimes_{\mathbb{Z}} \left(\mathbb{Z}/_4 \oplus \mathbb{Z}\right)$

$\cong \left(\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Z}/_4\right) \oplus \left(\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Z}\right) \oplus \left(\mathbb{Z}/_{17} \otimes_{\mathbb{Z}} \mathbb{Z}/_4\right) \oplus \left(\mathbb{Z}/_{17} \otimes_{\mathbb{Z}} \mathbb{Z}\right) \oplus \left(\mathbb{Z}/_2 \otimes_{\mathbb{Z}} \mathbb{Z}/_4\right) \oplus \left(\mathbb{Z}/_2 \otimes_{\mathbb{Z}} \mathbb{Z}\right)$

$\cong (\mathbb{Z}/_4)^2 \oplus \mathbb{Z}^2 \oplus 0 \oplus \mathbb{Z}/_{17} \oplus \mathbb{Z}/_2 \oplus \mathbb{Z}/_2$

$\cong \mathbb{Z}^2 \oplus \mathbb{Z}/_{17} \oplus (\mathbb{Z}/_4)^2 \oplus (\mathbb{Z}/_2)^2$

↗

Def.: A left R-module M is flat if $- \otimes_R M$ is exact.

Prop.: Projective modules are flat.

Proof: First note that a free R-module $F = \bigoplus_{i\in I} R$ is flat:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "j"] & B \arrow[r, "p"] & C \arrow[r] & 0
\end{tikzcd} \quad \text{exact}$$

$$\Big\downarrow \; - \otimes_R F$$

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_R F \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & B \otimes_R F \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C \otimes_R F \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
0 \arrow[r] & \bigoplus_{i\in I} A \arrow[r, "\oplus j"] & \bigoplus_{i\in I} B \arrow[r, "\oplus p"] & \bigoplus_{i\in I} C \arrow[r] & 0
\end{tikzcd} \quad \text{exact}$$

Let P be projective and $P \oplus P' \cong F$ free module. Then

$$0 \longrightarrow (A \otimes_R P) \oplus (A \otimes_R P') \longrightarrow (B \otimes_R P) \oplus (B \otimes_R P') \longrightarrow (C \otimes_R P) \oplus (C \otimes_R P') \longrightarrow 0$$

is exact iff each of the two individual sequences is exact. □

Prop.: Let $Q(R)$ be a quotient field of a PID R.

Then $Q(R)$ is a flat R-module.

Proof: [Bosch, Algebra, p. 304 (Chapter 7)]

Corollary: $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

Example: Let R be a commutative ring. Let $R[\Gamma]$ be the group ring of Γ over R.

(As a R-module, $R[\Gamma]$ is the free R-module over Γ.)

It has a ring multiplication: $\left(\sum_{\gamma\in\Gamma} r_\gamma \gamma\right) \cdot \left(\sum_{\gamma\in\Gamma} s_\gamma \gamma\right) = \sum_{\gamma\in\Gamma} \left(\sum_{\substack{\gamma_1,\gamma_2 \\ \gamma_1\gamma_2=\gamma}} r_{\gamma_1} s_{\gamma_2} \right) \gamma$. ↻

R is an $R[\Gamma]$-module via $\left(\sum_{\gamma\in\Gamma} r_\gamma \gamma\right)\cdot r = \sum_{\gamma\in\Gamma} r_\gamma r \in R$.

$$\begin{tikzcd} M/_{\langle\{\gamma m - m \mid \gamma\in\Gamma, m\in M\}\rangle} \arrow[r, "\cong"] & R \otimes_{R[\Gamma]} M \\ {[m]} \arrow[r, mapsto] & 1\otimes m \\ {[rm]} & r\otimes m \arrow[l, mapsto] \end{tikzcd}$$

Note that $1\otimes(\gamma m - m) = 1\cdot\gamma\otimes m - 1\otimes m = 1\otimes m - 1\otimes m = 0$;

so the map is well-defined.

Let C_2 be the group of order 2, $C_2 = \langle t\rangle$.

Then $\mathbb{Z}$ is a non-flat $\mathbb{Z}[C_2]$-module.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}[C_2] \arrow[r] & \mathbb{Z}[C_2] \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\ & x \arrow[r, mapsto] & x(1-t) & & \\ & & n+tm \arrow[r, mapsto] & n+m & \end{tikzcd}$$

Then $\mathbb{Z}\otimes_{\mathbb{Z}[C_2]} -$ gives

$$\begin{tikzcd} & & & \mathbb{Z} & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 0"] & \mathbb{Z} \arrow[r] & \mathbb{Z}\otimes_{\mathbb{Z}[C_2]}\mathbb{Z} \arrow[u, phantom, "\cong" rotate=90] \arrow[r] & 0 \end{tikzcd}$$

a sequence, which is <u>not</u> exact.

2.11.

6.5 Tor

Def.: Let M be a right R-module, N a left R-module. Let $P_* \to M$ be a projective resolution of M. We define the i-th Tor-group as

$$\mathrm{Tor}_i^R(M,N) := H_i(P_* \otimes_R N) \quad , \quad i \geq 0 .$$

Rem.: If $Q_* \to M$ is another ~~projectives~~ projective resolution, then id_M is induced by the fundamental theorem of homological algebra by a chain homotopy equivalence $P_* \to Q_*$ which is unique up to chain homotopy. So we obtain a preferred isomorphism $H_*(P_* \otimes_R N) \xrightarrow{\cong} H_*(Q_* \otimes_R N)$.

⌈ In particular, we have

$$\mathrm{Tor}_i^R(M,N) = \Big(\bigoplus_{P^* \text{ proj. res.}} H_i(P_* \otimes_R N) \Big) \Big/ \sim$$

where $\underset{H_i(P_*\otimes_R N)}{\overset{\cap}{x}} \sim \underset{H_i(P'_*\otimes_R N)}{\overset{\cap}{y}}$ if ~~theig~~ they correspond to each other under the preferred isomorphism $H_i(P_* \otimes_R N) \xrightarrow{\cong} H_i(P' \otimes_R N)$. ⌋

Rem.: $\mathrm{Tor}_i^R(-,-)$ is a (covariant) functor in both variables. Clearly, a homomorphism $g: N \to N'$ induces a chain map $\mathrm{id}_* \otimes g : P_* \otimes_R N \to P_* \otimes_R N'$, hence a homomorphism of abelian groups: $\mathrm{Tor}_i^R(\mathrm{id}, g) : \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M,N')$.
Let $f: M \to M'$ be a homomorphism and P'_* be a projective resolution of M'. ↺

Then $f = H_0(f_*)$ for a chain map $f_* : P_* \to P'_*$ which induces $f_* \otimes id_N : P_* \otimes_R N \to P'_* \otimes_R N$ and $Tor_i^R(f, id) : Tor_i^R(M,N) \to Tor_i^R(M', N$

We state without proof:

<u>Thm.</u> [Lück, lemma 6.12]:

1) If Q_* is a projective resolution of N, then $Tor_i^R(M,N) \cong H_i(M \otimes_R Q_*$ $i \geq 0$.

2) Let R be a commutative ring. Then there is a natural isomorphism $$Tor_i^R(M,N) \cong Tor_i^R(N,M), \quad i \geq 0.$$

<u>Proposition:</u> There is a natural isomorphism
$$Tor_0^R(M,N) \cong M \otimes_R N.$$

<u>Proof:</u> Let $P_* \to M$ be a projective resolution. Since $- \otimes_R N$ is right exact, $P_1 \otimes_R N \to P_0 \otimes_R N \to M \otimes_R N \to 0$ is exact. Hence $M \otimes_R N \cong \operatorname{coker}(P_1 \otimes_R N \to P_0 \otimes_R N)$ $\square$

$Tor_i^R(-,-)$ measures the defect of left exactness of tensor products.

<u>Thm.</u> (LES for Tor):

1) Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be a short exact sequence of right R-modules. Let N be a left R-module. Then there is a natural long exact sequence $\ldots \to Tor_2^R(M_0,N) \to Tor_2^R(M_1,N) \to Tor_2^R(M_2,N) \to Tor_1^R(M_0,N) \to$
$$\to Tor_1^R(M_1,N) \to Tor_1^R(M_2,N) \to M_0 \otimes_R N \to M_1 \otimes_R N \to M_2 \otimes_R N \to 0.$$

2) Let $0 \to N_0 \to N_1 \to N_2 \to 0$ be a short exact sequence of left R-modules. Let M be a right R-module. Then there is a natural long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(M, N_0) \to \mathrm{Tor}_1^R(M, N_1) \to \mathrm{Tor}_1^R(M, N_2)$$
$$\to M \otimes_R N_0 \to M \otimes_R N_1 \to M \otimes_R N_2 \to 0 .$$

Lemma: Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be a short exact sequence of right R-modules. There are projective resolutions

$$P_*^{(i)} \to M_i \to 0 \quad , \quad i \in \{0, 1, 2\}$$

and chain maps $P_*^{(0)} \to P_*^{(1)}$ and $P_*^{(1)} \to P_*^{(2)}$ s.th.

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M_0 \arrow[r] \arrow[u] & M_1 \arrow[r] \arrow[u] & M_2 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_0^{(0)} \arrow[r] \arrow[u] & P_0^{(1)} \arrow[r] \arrow[u] & P_0^{(2)} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_1^{(0)} \arrow[r] \arrow[u] & P_1^{(1)} \arrow[r] \arrow[u] & P_1^{(2)} \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

commutes and has exact rows.

Proof: Choose projective resolutions $P_*^{(0)}$ and $P_*^{(2)}$ first. We then construct $P_*^{(1)}$ and the chain maps inductively.

Step 0:

$$\begin{tikzcd}
 & 0 & & 0 & \\
0 \arrow[r] & M_0 \arrow[r] \arrow[u] & M_1 \arrow[r] & M_2 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_0^{(0)} \arrow[r, "j_0"] \arrow[u, "\varepsilon^{(0)}"] & P_0^{(1)} \arrow[r, "p_0"] \arrow[u, "\varepsilon^{(1)}"] & P_0^{(2)} \arrow[r] \arrow[u, "\varepsilon^{(2)}"'] & 0
\end{tikzcd}$$

Set $P_0^{(1)} := P_0^{(0)} \oplus P_0^{(2)}$ and j_0 to be the inclusion and p_0 to be the projection. ↺

Define $\varepsilon^{(1)}$ by $\varepsilon^{(1)}|_{P_0^{(0)}} = j \circ \varepsilon^{(0)}$ and $\varepsilon^{(1)}|_{P_0^{(2)}} = \overline{\varepsilon^{(2)}}$ where $\overline{\varepsilon^{(2)}}$ is the solution of

$$\begin{tikzcd}
M_1 \arrow[r, "p"] & M_2 \arrow[r] & 0 \\
& P_0^{(2)} \arrow[u, "\varepsilon^{(2)}"'] \arrow[ul, dashed, "\overline{\varepsilon^{(2)}}"'] &
\end{tikzcd}$$

That $\varepsilon^{(1)}$ is surjective follows from a diagram chase.

Now you continue in a similar fashion.

Proof of the LES for Tor:

1) Choose $P_*^{(i)} \to M_i$ as in the previous lemma. Since

$$0 \to P_*^{(0)} \to P_*^{(1)} \to P_*^{(2)} \to 0$$

splits degree-wise,

$$(*) \quad 0 \to P_*^{(0)} \otimes_R N \to P_*^{(1)} \otimes_R N \to P_*^{(2)} \otimes_R N$$

is an exact sequence of chain complexes.

The LES of $(*)$ yields the LES for Tor.

2) Let P_* be a projective resolution of M.

Then

$$(**) \quad 0 \to P_* \otimes_R N_0 \to P_* \otimes_R N_1 \to P_* \otimes_R N_2 \to 0$$

is exact since projective modules are flat.

The LES of $(**)$ yields the LES for Tor. ▨

Remark: $\mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, M) = H_i(G; M)$ is called the group homology of G with coefficients in M which is of great ↗ importance in group theory. See e.g. Brown's "Cohomology of groups".

Rem.: 1) If M is projective, then

$$\mathrm{Tor}_i^R(M,N) = 0 \qquad \forall i>0,$$

since $0 \to P_0 \xrightarrow{id} M \to 0$ is a projective resolution.

2) If K is a field, then $\mathrm{Tor}_i^K(M,N)=0 \quad \forall i>0$, since every K-module is free.

Prop.: Tor commutes with arbitrary direct sums.

Proof: Follows immediately from the fact that tensor products commute with direct sums. ▨

Prop.: (Tor for PID's)

Let R be a PID. Then the following holds:

1) For all R-modules M, N, $\mathrm{Tor}_i^R(M,N) = 0$ for $i \geq 2$.

2) The inclusions $\mathrm{tors}(M) \hookrightarrow M$ and $\mathrm{tors}(N) \hookrightarrow N$ of torsion submodules into finitely generated R-modules M and N induce an isomorphism

$$\mathrm{Tor}_i^R(\mathrm{tors}(M), \mathrm{tors}(N)) \xrightarrow{\cong} \mathrm{Tor}_i^R(M,N) .$$

3) Let $r,s \in R\setminus\{0\}$. Let $d \in R$ be such that for the generated ideals we have $(r,s)=(d)$ (so $\gcd(r,s)=d$). Then

$$\mathrm{Tor}_1^R(R/_{(r)}, S/_{(s)}) \cong R/_{(d)} .$$

↺

Proof: Ad 1): Take an epimorphism $F \xrightarrow{p} M \longrightarrow 0$ from a free R-module F to M. Then $\ker(p) \subset F$ is free as a submodule of a free R-module over a PID R.
(see Rotman, Advanced modern algebra, thm. 8.9)

$$\rightsquigarrow \quad 0 \longrightarrow \ker(p) \longrightarrow F \xrightarrow{p} M \longrightarrow 0$$

is a projective resolution implying 1).

Ad 2): $M = \underset{}{\mathrm{tors}(M)} \oplus \underset{\text{free}}{F_M}$, $N = \mathrm{tors}(N) \oplus \underset{\text{free}}{F_N}$. 3.11.

The claim follows from the fact that Tor commutes with $\oplus$ and vanishes if one of the input modules is free.

Ad 3): Consider $0 \longrightarrow R \xrightarrow{r} R \longrightarrow R/_{(r)} \longrightarrow 0$ exact.

$$\mathrm{LES} \Rightarrow \mathrm{Tor}_1^R\left(R/_{(r)}, R/_{(s)}\right) \cong \ker\left(R/_{(s)} \xrightarrow{\cdot r} R/_{(s)}\right) \xleftarrow{\cong} R/_{(d)}$$
$$s_d \cdot x \longleftarrow\!\shortmid \; x$$

where $s = s_d \cdot d$. ☒

6.6 The universal coefficient theorem for homology.

Thm.: Let R be a PID. Let C_* be a projective positive chain complex of R-modules (C_n projective, $C_n = 0$ for $n<0$). Let M be an R-module. Then there is a short exact sequence

$$0 \longrightarrow H_n(C_*) \otimes_R M \xrightarrow{\alpha_n} H_n(C_* \otimes_R M) \xrightarrow{\beta_n} \mathrm{Tor}_1^R(H_{n-1}(C_*), M) \longrightarrow 0$$
$$[c] \otimes m \longmapsto [c \otimes m]$$

for every $n \geq 0$. The maps α_n and β_n are natural in C_* and M. Moreover, the sequence splits via s_n: $\mathrm{Tor}_1^R(H_{n-1}(C_*), M) \to H_n(C_* \otimes M)$ ↗

which is natural in M but, in general, not natural in C_*.
In particular, the map

$$\alpha_n \oplus s_n : \; H_n(C_* \otimes_R M) \oplus \mathrm{Tor}_1^R(H_{n-1}(C_*), M) \xrightarrow{\cong} H_n(C_* \otimes_R M)$$

is an isomorphism which is natural in M but not in C_*.

Proof: Denote by $Z_n \subset C_n$ cycles and
$B_n \subset C_n$ boundaries in C_*.

Consider the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, "i_n"] & C_n \arrow[r, "c_n"] & B_{n-1} \arrow[l, dashed, bend left] \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & B_n \arrow[r, "j_n"] & Z_n \arrow[r, "p_n"] & H_n(C_*) \arrow[r] & 0 \end{tikzcd}.$$

(see proof of
Euler-Poincaré formul

Since R is a PID, Z_n and B_n are projective for all n.
Thus

$$0 \longrightarrow Z_n \otimes_R M \longrightarrow C_n \otimes_R M \longrightarrow B_{n-1} \otimes_R M \longrightarrow 0$$

is exact.

Now we regard Z_* and B_{*-1} as chain complexes with zero differentials.(1) Then

$$0 \longrightarrow Z_* \otimes_R M \longrightarrow C_* \otimes_R M \longrightarrow B_{*-1} \otimes_R M \longrightarrow 0 \qquad (*)$$

is a short exact sequence of chain complexes.
Next consider the LES associated to $(*)$:

$$\longrightarrow \ldots \longrightarrow B_n \otimes_R M \xrightarrow{\partial_{n+1}} Z_n \otimes_R M \longrightarrow \boxed{H_n(C_* \otimes_R M)} \longrightarrow \qquad (**)$$

$$\longrightarrow B_{n-1} \otimes_R M \xrightarrow{\partial_n} Z_{n-1} \otimes_R M \longrightarrow \quad \ldots$$

The boundary homomorphisms ∂_* are induced by the inclusion of boundaries into cycles.

↺

---

(1) Therefore, they usually don't appear as chain complexes of spaces.

Tensoring the second short exact sequence by M yields

$$\mathrm{Tor}_1^R(H_n(C_*), M) \longrightarrow B_n \otimes_R M \xrightarrow{\partial_{n+1}} Z_n \otimes M \longrightarrow H_n(C_*) \otimes_R M \longrightarrow 0 \qquad (***)$$

Splitting $(**)$ into short exact sequences gives:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker}(\partial_{n+1}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_n(C_* \otimes_R M) \arrow[r] & \ker(\partial_n) \arrow[r] \arrow[d, phantom, "\cong\ \text{by } (***)"] & 0 \\
 & H_n(C_*) \otimes_R M & & \mathrm{Tor}_1^R(H_{n-1}(C_*), M) &
\end{tikzcd}$$

by right exactness of $\otimes_R M$

The sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$ splits.

Let $C_n \xrightarrow{r_n} Z_n$ be a left inverse of the inclusion $Z_n \hookrightarrow C_n$.

Define $H_n(C_* \otimes_R M) \longrightarrow H_n(C_*) \otimes_R M$.

$$[c \otimes m] \longmapsto [r_n(c) \otimes m]$$

It is obviously a left inverse of α_n. ☒

Now apply the previous theorem to the singular chain complex $C_*^{\mathrm{sing}}(X, A; R)$. Note that:

$$C_*^{\mathrm{sing}}(X, A; R) = C_*^{\mathrm{sing}}(X, A) \otimes_{\mathbb{Z}} R.$$

We might also consider homology with coefficients in an R-module M:

$$\begin{aligned} C_*^{\mathrm{sing}}(X, A; M) &= C_*^{\mathrm{sing}}(X, A) \otimes_{\mathbb{Z}} M \\ &= C_*^{\mathrm{sing}}(X, A; R) \otimes_R M \end{aligned}$$

and $H_*^{\mathrm{sing}}(X, A; M) := H_*(C_*^{\mathrm{sing}}(X, A; M))$. So we obtain: ↗

Theorem (universal coefficient theorem for homology):

Let R be a PID. Let M be an R-module and (X,A) a pair of spaces. Then there is a short exact sequence

$$0 \longrightarrow H_n(X,A;R)\otimes_R M \longrightarrow H_n(X,A;M) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(X,A;R),M) \longrightarrow 0$$

$$[x]\otimes m \longmapsto [x\otimes m]$$

natural in (X,A) and in M which splits naturally in M.

The most important case is $R=\mathbb{Z}$, $M=\mathbb{Z}/n$.

6.7 Ext

A cochain complex C^* of R-modules is a sequence $(C^n)_{n\in\mathbb{Z}}$ of R-modules together with $c^n: C^n \to C^{n+1}$ s.th. $c^{n+1}\circ c^n = 0$ $\forall n$.

[That is, $C_* := C^{-*}$ is a chain complex.]

If C_* is a chain complex of left R-modules and M is a left R-module, then $\hom_R(C_*,M)$ is a cochain complex of abelian groups with the induced differentials

$$\hom_R(C_n,M) \longrightarrow \hom_R(C_{n+1},M).$$
$$f \longmapsto f\circ c_{n+1}$$

Terminology / notation:

Z^n = Cocycles = $\ker(c^n: C^n \to C^{n+1})$

B^n = Coboundaries = $\mathrm{im}\,(c^{n-1}: C^{n-1} \to C^n)$

A generic notation for differentials in chain complexes and cochain complexes is ∂_* and δ^*, respectively.

↺

The <u>cohomology</u> $H^n(C^*)$ of a cochain complex C^* is defined as $H^n(C^*) = \frac{\ker(c^n)}{\operatorname{im}(c^{n-1})}$.

<u>Def.</u>: Let M, N be left R-modules. Let P_* be a projective resolution of M. We define

$$\operatorname{Ext}^i_R(M,N) = H^i(\hom_R(P_*,N)) \quad , \quad i \geq 0 .$$

<u>Remark</u>: • One can prove that

$$\operatorname{Ext}^i_R(M,N) \cong H^i(\hom_R(M, I^*))$$

for an injective resolution I^* of N.

- Like for Tor, one proves functoriality of Ext and independance of P_*.

<u>Thm</u> (LES for Ext):

Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be a short exact sequence of left R-modules. Let N be a left R-module.

Then there are two natural long exact sequences

$$\text{(i)} \quad 0 \to \hom_R(N,M_0) \to \hom_R(N,M_1) \to \hom_R(N,M_2) \to$$
$$\to \operatorname{Ext}^1_R(N,M_0) \to \operatorname{Ext}^1_R(N,M_1) \to \operatorname{Ext}^1_R(N,M_2) \to$$
$$\to \operatorname{Ext}^2_R(N,M_0) \to \cdots$$

$$\text{(ii)} \quad 0 \to \hom_R(M_2,N) \to \hom_R(M_1,N) \to \hom_R(M_0,N) \to$$
$$\to \operatorname{Ext}^1_R(M_2,N) \to \operatorname{Ext}^1_R(M_1,N) \to \operatorname{Ext}^1_R(M_0,N) \to$$
$$\to \operatorname{Ext}^2_R(M_2,N) \to \cdots$$

↗

Proof: Analogous to the proof of the corresponding statement for Tor. ☒

Proposition: We have a natural isomorphism

$$\mathrm{Ext}_R^0(M,N) \cong \hom_R(M,N)$$

for all R-modules M, N.

Proof: Similar proof as for Tor. ☒

9.11.

Proposition: P is projective $\iff$ $\mathrm{Ext}_R^i(P,N)=0$ for $i \geq 1$ and every left R-module N.

Proof: "$\Rightarrow$": clear ✓

"$\Leftarrow$": $\mathrm{Ext}_R^1(P,N)=0$ for all N implies via LES that $\hom_R(P,-)$ is exact, thus P is projective. ☒

Proposition: There are nat. isomorphisms

$$\mathrm{Ext}_R^i\Big(\bigoplus_{j\in J} M_j, N\Big) \cong \prod_{j\in J} \mathrm{Ext}_R^i(M_j,N)$$

$$\mathrm{Ext}_R^i\Big(M, \prod_{j\in J} N_j\Big) \cong \prod_{j\in J} \mathrm{Ext}_R^i(M,N_j) .$$

Proof: Follows from the corresponding properties of hom:

$$\mathrm{Ext}_R^i\Big(\bigoplus_{j\in J} M_j, N\Big) = H^i\Big(\hom_R\Big(\bigoplus_{j\in J} P_*^{(j)}, N\Big)\Big) = H^i\Big(\prod_{j\in J}\hom_R(P_*^{(j)},N)\Big)$$

$$= \prod_{j\in J} \underbrace{H^i\big(\hom_R(P_*^{(j)},N)\big)}_{\mathrm{Ext}_R^i(M_j,N)} .$$

Similarly for 2nd property. ☒ ↺

Remark: $\operatorname{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, N) = H^i(G, N)$ is called the group cohomology of G with coefficients in N.

Thm.: (universal coefficient theorem for cochain complexes)

Let R be a PID. Let C_* be a positive projective R-chain complex. Let M be an R-module.

Then there is a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ext}^1_R(H_{n-1}(C_*), M) \arrow[r, "\beta^n"] & H^n(\hom_R(C_*, M)) \arrow[r, "\alpha^n"] \arrow[l, dashed, bend left, "s^n"] & \hom_R(H_n(C_*), M) \arrow[r] & 0 \end{tikzcd}$$

where $\alpha^n([f: C_n \to M]) = \left\{ \begin{array}{c} H_n(C_*) \to M \\ \overset{\Psi}{[x]} \mapsto f(x) \end{array} \right\}$.

Moreover, the sequence splits with a homomorphism s^n as above, and α^n, β^n are natural in C_* and M and s^n is only natural in M.

6.8 Künneth theorems

In this section, R is a commutative ring.

Def.: Let C_* and D_* be R-chain complexes. The tensor product of C_* and D_* is a R-chain complex $C_* \otimes_R D_*$ with

$$\begin{cases} (C_* \otimes_R D_*)_n := \bigoplus_{p+q=n} C_p \otimes_R D_q \\ \partial^{\otimes}(\underset{\substack{\Uparrow \\ (C_* \otimes_R D_*)_n}}{x \otimes y}) = \partial x \otimes y + (-1)^{\deg x} x \otimes \partial y \in (C_* \otimes_R D_*)_{n-1}. \end{cases}$$

Note that $\partial^{\otimes}\partial^{\otimes}(x \otimes y) = \partial^{\otimes}(\partial x \otimes y + (-1)^{\deg x} x \otimes \partial y) = \underbrace{\partial\partial x \otimes y}_{=0} + (-1)^{\deg x - 1} \partial x \otimes \partial y$
$+ (-1)^{\deg x}(\partial x \otimes \partial y + (-1)^{\deg x} \otimes \partial\partial y) = 0$.

↗

In the next section we prove the following thm.:

Theorem: (Eilenberg – Zilber theorem)

1) There is – up to natural chain homotopy – a unique chain map

$$C_*^{\text{sing}}(X\times Y;R) \longrightarrow C_*^{\text{sing}}(X;R) \otimes_R C_*^{\text{sing}}(Y;R)$$

that maps a singular 0-simplex $(x,y) \in X\times Y$ to $x \otimes y \in C_0^{\text{sing}}(X;R) \otimes_R C_0^{\text{sing}}(Y;R)$.

2) There is – up to natural chain homotopy – a unique chain map

$$C_*^{\text{sing}}(X;R) \otimes_R C_*^{\text{sing}}(Y;R) \longrightarrow C_*^{\text{sing}}(X\times Y;R)$$

that maps $x \otimes y$ (where x and y are regarded as 0-simplices) to $(x,y) \in C_0^{\text{sing}}(X\times Y;R)$.

Def.: The Alexander – Whitney map

$$AW_*: C_*^{\text{sing}}(X\times Y;R) \longrightarrow C_*^{\text{sing}}(X;R) \otimes_R C_*^{\text{sing}}(Y;R)$$

$$AW_n((\sigma,\tau)) = \sum_{i=0}^{n} \sigma|_{[e_1,\dots,e_{i+1}]} \otimes \tau|_{[e_{i+1},\dots,e_{n+1}]}$$

is a map as in 1).

Every chain map as in 2) is called Eilenberg–Zilber map and generically denoted by EZ_*.

Cor.: AW_* and EZ_* yield natural homotopy equivalences

$$C_*^{\text{sing}}(X\times Y;R) \xrightarrow{\simeq} C_*^{\text{sing}}(X;R) \otimes_R C_*^{\text{sing}}(Y;R).$$

↺

Remark: Let X and Y be CW-complexes (assume further that e.g. X is locally compact).

Then $X \times Y$ is a CW complex via

$$(X \times Y)^n = \bigcup_{p+q=n} X^p \times Y^q .$$

So the n cells of $X \times Y$ are

$$\left\{ e^i \times \tilde{e}^{n-i} \mid i \geq 0 ,\ e^i \text{ i-cell in } X,\ \tilde{e}^{n-i} \text{ (n-i)-cell in } Y \right\} .$$

Thus

$$C_n^{CW}(X \times Y; R) \cong \bigoplus_{i=0}^{n} C_i^{CW}(X;R) \otimes_R C_{n-i}^{CW}(Y;R) .$$

This iso holds as chain complexes (see Hatcher p. 269).

Next we deduce a relative version of the EZ chain homotopy equivalence. Recall from section 2.2 the notion of <u>excisive triad</u> $(X; A, B)$:

$(X; A, B)$ excisive iff $(A, A \cap B) \hookrightarrow (X, B)$ induces a homology iso.

Let $(X; A, B)$ be excisive s.th. $X = A \cup B$.

By applying the 5-lemma and LES to

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
C_*^{sing}(B) \arrow[rr, "id"] \arrow[d] & & C_*^{sing}(B;R) \arrow[d] \\
C_*^{sing}(A;R) + C_*^{sing}(B;R) \arrow[rr] \arrow[d] & & C_*^{sing}(A \cup B;R) \arrow[d] \\
\left(C_*^{sing}(A;R) + C_*^{sing}(B;R) \right) / C_*^{sing}(B) \arrow[rr, "H_*\text{-iso}"] \arrow[d] & & C_*^{sing}(A \cup B;R) / C_*^{sing}(B;R) \arrow[d] \\
{} & & {}
\end{tikzcd} ,$$

$\cong C_*^{sing}(A;R) / C_*^{sing}(A \cap B;R)$

↗

this shows that $(A\cup B; A, B)$ being excisive is equivalent to

$$C_*^{sing}(A;R) + C_*^{sing}(B;R) \longrightarrow C_*^{sing}(A\cup B; R)$$

being a homology isomorphism.

Def.: We call a pair of subspaces (A,B) excisive if $(A\cup B, A, B)$ is excisive.

Prop.: (A,B) is excisive if one of the following properties holds:

1) $A, B \subset A\cup B$ open
2) $A, B \subset A\cup B$ closed and neighborhood deformation retracts
3) A, B subcomplexes of $A\cup B$ (CW complex)

Proof: 1) follows from the "$\mathcal{U}$-small simplices lemma"
2) can be reduced to 1), similar considerations last semester
3) follows from 2). ☒

Thm.: Let (X,A) and (Y,B) pairs of spaces s.th. $(X\times B, A\times Y)$ is excisive. Then there is a natural isomorphism for $n\geq 0$:

$$\begin{tikzcd} H_n(C_*^{sing}(X,A;R) \otimes_R C_*^{sing}(Y,B;R)) \arrow[d, "\cong"] \\ \underbrace{H_n\left(C_*^{sing}(X\times Y, A\times Y \cup X\times B; R)\right)}_{H_n((X,A)\times(Y,B);R)} \end{tikzcd}$$

We write $(X,A)\times(Y,B) := A\times Y \cup X\times B$.

↺

Proof:

$$\begin{tikzcd}
C_*^{sing}(X\times Y;R)\Big/ C_*^{sing}(A\times Y;R)+C_*^{sing}(X\times B;R) \arrow[d] \\
C_*^{sing}\big((X,A)\times(Y,B);R\big) = C_*^{sing}(X\times Y;R)\Big/ C_*^{sing}(A\times Y\cup X\times B;R)
\end{tikzcd}$$

is a homology iso by excisiveness and 5-lemma + LES.

Now consider:

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix} C_*^{sing}(X;R)\otimes_R C_*^{sing}(B;R) \\ + \\ C_*^{sing}(A;R)\otimes_R C_*^{sing}(Y;R) \end{matrix} \arrow[r] \arrow[d, "\text{ind. } AW_*"', "\simeq"] & C_*^{sing}(X;R)\otimes C_*^{sing}(Y;R) \arrow[r] \arrow[d, "\simeq"', "AW_*"] & C_*^{sing}(X,A;R)\otimes_R C_*^{sing}(Y,B;R) \arrow[r] \arrow[d, "\begin{matrix}\text{5-lemma + LES}\\ \Rightarrow H_*\text{-iso}\end{matrix}"] & 0 \\
0 \arrow[r] & \begin{matrix} C_*^{sing}(X\times B;R) \\ + \\ C_*^{sing}(A\times Y;R) \end{matrix} \arrow[r] & C_*^{sing}(X\times Y;R) \arrow[r] & C^{sing}(X\times Y;R)\Big/ C_*^{sing}(A\times Y;R)+C_*^{sing}(X\times B;R) \arrow[r] & 0
\end{tikzcd}$$

$\boxtimes$

Theorem (Künneth theorem for chain complexes):

Let R be a PID. Let C_* and D_* be positive R-chain complexes, and let C_n be projective for all n. Then there is a natural short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(C_*) \underset{R}{\otimes} H_j(D_*) \longrightarrow H_n(C_* \underset{R}{\otimes} D_*) \longrightarrow \bigoplus_{i+j=n-1} \operatorname{Tor}^1_R(H_i(C_*), H_j(D_*)) \longrightarrow 0$$

which (not naturally) splits.

Proof: First consider the case where C_* has vanishing differentials. Then for the differentials of $C_* \underset{R}{\otimes} D_*$ we have

$$\partial(c \otimes d) = \partial c \otimes d + (-1)^{\deg c} \cdot c \otimes \partial d = \pm\, c \otimes \partial d \,.$$

Hence $C_* \underset{R}{\otimes} D_*$ is (up to sign) a direct sum of the chain complexes

$$C_i \underset{R}{\otimes} D_{*-i} \,.$$

$$\Rightarrow \quad H_n(C_* \underset{R}{\otimes} D_*) = H_n\Big(\bigoplus_i C_i \underset{R}{\otimes} D_{*-i}\Big) \cong \bigoplus_i H_n(C_i \underset{R}{\otimes} D_{*-i})$$

$$\underset{C_i \text{ projective}}{\cong} \bigoplus_i C_i \underset{R}{\otimes} H_n(D_{*-i})$$

$$= \bigoplus_i C_i \underset{R}{\otimes} H_{n-i}(D_*) \,.$$

This proves the result in this special case.

Now let C_* be arbitrary. Consider

$$\begin{tikzcd} & \text{cycles + zero diff's} & & \text{boundaries + zero diff's} & \\ 0 \arrow[r] & Z_* \arrow[r] & C_* \arrow[r] & B_{*-1} \arrow[r] & 0 \,. \\ & \text{projective, since } R \text{ PID} & & \text{projective, since } R \text{ PID} & \end{tikzcd}$$

↺

$$\rightsquigarrow \quad 0 \longrightarrow Z_* \otimes_R D_* \longrightarrow C_* \otimes_R D_* \longrightarrow B_{*-1} \otimes_R C_* \longrightarrow 0 \quad \text{exact}$$

The boundary homomorphism of the associated LES is induced by the inclusion $B_{*-1} \subset Z_*$.

$$\begin{tikzcd} {} \arrow[r, "\overline{i_n}"] & H_n(Z_* \otimes_R D_*) \arrow[r] \arrow[d, phantom, "\cong"] & H_n(C_* \otimes_R D_*) \arrow[r] & H_n(B_{*-1} \otimes_R D_*) \arrow[r] \arrow[d, phantom, "\cong"] & \cdots \\ & \bigoplus_{p+q=n} Z_p \otimes H_q(D_*) & & \bigoplus_{p+q=n} B_{p-1} \otimes_R H_q(D_*) & \\ {} \arrow[r, "\overline{i_{n-1}}"] & H_{n-1}(Z_* \otimes_R D_*) \arrow[r] & \cdots & & \end{tikzcd}$$

$$\begin{tikzcd} \rightsquigarrow \quad 0 \arrow[r] & \operatorname{coker}(\overline{i_n}) \arrow[r] \arrow[d, phantom, "\cong"] & H_n(C_* \otimes_R D_*) \arrow[r] & \ker(\overline{i_{n-1}}) \arrow[r] & 0 \quad \text{exact} \\ & \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*) & & & \end{tikzcd}$$

right exactness of $\_ \otimes_R H_q(D_*)$

The iso $\ker(\overline{i_{n-1}}) \cong \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_*), H_q(D_*))$ results from "summing up"

$$0 \longrightarrow \mathrm{Tor}^1_R(H_p(C_*), H_q(D_*)) \longrightarrow B_p \otimes_R H_q(D_*) \longrightarrow Z_p \otimes_R H_q(D_*) \longrightarrow 0$$

(universal coefficient theorem) $\square$

Theorem (Künneth for homology of spaces):

Let R be a PID. Let (X,A) and (Y,B) be pairs of spaces s.th. $(X\times B, A\times Y)$ is excisive (e.g. $A=B=\emptyset$ or (X,A) and (Y,B) are CW pairs.). There is a natural short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X,A;R)\underset{R}{\otimes} H_q(Y,B;R) \xrightarrow{\times} H_n((X,A)\times(Y,B);R) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X,A;R), H_q(Y,B;$$
$$\longrightarrow 0$$

Corollary: Let k be a field. Then we get

$$H_n(X\times Y;k) \cong \bigoplus_{p+q=n} H_p(X;k)\underset{R}{\otimes} H_q(Y;k).$$

Remark: The map $\times$ is induced by AW_* followed by H_*-iso from excisiveness.

Proof: follows immediately from the Künneth thm for chain complexes and the (relative) EZ chain homotopy equivalence. ☒

6.9 The method of acyclic models

Let $\mathcal{C}$ be a category (small). Let R be a commutative ring.

Def.: An $R\mathcal{C}$-module is a functor $\mathcal{C}\longrightarrow R\text{-Mod}$.

Example: $\mathcal{C}=G$ group.

What is an $R\mathcal{C}$-module then? – It's an R-module M with a homomorphism $G\xrightarrow{\varphi}\mathrm{Aut}_R(M)$. This is the same as the structure of an $R[G]$-module on M:

$$\Big(\underset{\substack{\Uparrow\\ R[G]}}{\sum r_g\cdot g}\Big)\cdot m = \sum r_g\,\varphi(g)(m)\in M$$

A homomorphism of $R\mathcal{C}$-modules is a natural transformation. ↺

The category of $R\mathcal{C}$-modules is an abelian category – for us, this means that we can do homological algebra in this category.

For example, a sequence of $R\mathcal{C}$-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is <u>exact</u> if

$$0 \to M_1(c) \to M_2(c) \to M_3(c) \to 0$$

is exact as R-modules for any object $c \in ob(\mathcal{C})$.

One defines <u>projectivity</u> of $R\mathcal{C}$-modules by the universal property

$$\begin{tikzcd} M \arrow[r] & N \arrow[r] & 0 \\ & P \arrow[u] \arrow[ul, dashed] & \end{tikzcd}$$

.

<u>Remark</u>: $M(c)$ projective for all $c \in ob(\mathcal{C}) \not\Rightarrow M$ projective

<u>Def.</u>: Let $(c_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. Then the $R\mathcal{C}$-module

$$F = \bigoplus_{i \in I} R[mor_{\mathcal{C}}(c_i, -)] \qquad \left(\begin{array}{c} ob(\mathcal{C}) \ni c \mapsto \bigoplus_{i\in I} R[mor_{\mathcal{C}}(c_i, c)] \\ \text{on objects} \end{array} \right)$$

is called <u>free</u>. The family (c_i) are the <u>models</u> of F.

<u>Proposition</u>: Free $R\mathcal{C}$-modules are projective.

<u>Proof</u>: It is enough to prove that $F = R[mor_{\mathcal{C}}(c, -)]$ is projective. Consider:

$$\begin{tikzcd} M \arrow[r, "q"] & N \arrow[r] & 0 \\ & F \arrow[u, "f"'] \arrow[ul, dashed, bend left, "\bar{f}"] & \end{tikzcd}$$

First consider
$$\begin{tikzcd} \mathcal{M}(c) \arrow[r, "q(c)"] & N(c) \arrow[r] & 0 \\ & F(c) = R[\mathrm{mor}_{\mathcal{C}}(c,c)] \arrow[u, "f(c)"'] & \end{tikzcd}$$

Let $m_0 \in \mathcal{M}$ be a $q(c)$-preimage of $f(c)(\mathrm{id}_{\mathcal{C}}) \in N(c)$.

Set $\overline{f}(c)(\mathrm{id}_c) := m_0$.

For $(c \xrightarrow{\varphi} d) \in \mathrm{mor}_{\mathcal{C}}(c,d) \subseteq R[\mathrm{mor}_{\mathcal{C}}(c,d)] = F(d)$

we define (forced by naturality)

$$\begin{aligned} \overline{f}(d)(c \xrightarrow{\varphi} d) &= \overline{f}(d)(c \xrightarrow{\mathrm{id}} c \xrightarrow{\varphi} d) \\ &= \mathcal{M}(\varphi)\big(\underbrace{\overline{f}(c)(c \xrightarrow{\mathrm{id}} c)}_{\mathcal{M}(c)}\big) \in \mathcal{M}(d) \\ &= \mathcal{M}(\varphi)(m_0). \end{aligned}$$

One easily checks that $\overline{f}$ defined as above is a natural transformation. ▨

Example: $\quad C_n^{\mathrm{sing}}(-;R) : \underset{\overset{\shortparallel}{\mathcal{C}}}{\mathrm{Top}} \longrightarrow R\text{-}\mathcal{M}od$

is a free $R\,\mathrm{Top}$-module: $\quad C_n^{\mathrm{sing}}(-;R) = R[\mathrm{mor}_{\mathrm{Top}}(\Delta_n, -)]$

$C_*^{\mathrm{sing}}(-;R)$ is a free $R\mathcal{C}$-chain complex.

The following theorem has essentially the same proof as for R-modules:

↺

<u>Thm.:</u> Let F_* be a projective, positive $R\mathcal{C}$-chain complex. Let G_* be a positive $R\mathcal{C}$-chain complex such that

$$H_n(G_*(c)) = 0 \quad \text{for all } c \in ob(\mathcal{C}),\ n > 0.$$

There is a bijection

$$[F_*, G_*] \xrightarrow{\sim} \hom_{R\mathcal{C}}(H_0 \circ F_*, H_0 \circ G_*).$$

<u>Thm.</u> (acyclic model theorem):

Let F_* be a positive free $R\mathcal{C}$-chain complex. Let $\mathcal{M} \in ob(\mathcal{C})$ be the set of models of all F_n. Let G_* be an $R\mathcal{C}$-chain complex s.th.

$$H_n(G_*(c)) = 0 \quad \text{for all } c \in \overset{M}{\cancel{ob(\mathcal{C})}},\ n > 0.$$

Then

$$[F_*, G_*] \xrightarrow{\cong} \hom_{R\mathcal{C}}(H_0 \circ F_*, H_0 \circ G_*).$$

<u>Proof of EZ chain homotopy equivalence:</u>

Let $\mathcal{C} = \mathrm{Top} \times \mathrm{Top}$. Consider the two $R\mathcal{C}$-chain complexes

$$F_* = C_*^{sing}(- \times -; R), \quad G_* = C_*^{sing}(-;R) \otimes_R C_*^{sing}(-;R).$$

First consider
$$\begin{tikzcd} \mathcal{M}(c) \arrow[r, "q(c)"] & \mathcal{N}(c) \arrow[r] & 0 \\ & \mathcal{F}(c) = R[\mathrm{mor}_{\mathcal{C}}(c,c)] \arrow[u, "f(c)"] & \end{tikzcd}$$
.

Let $m_0 \in \mathcal{M}$ be a $q(c)$-preimage of $f(c)(\mathrm{id}_c) \in \mathcal{N}(c)$.

Set $\bar{f}(c)(\mathrm{id}_c) := m_0$.

For $(c \xrightarrow{\varphi} d) \in \mathrm{mor}_{\mathcal{C}}(c,d) \subseteq R[\mathrm{mor}_{\mathcal{C}}(c,d)] = \mathcal{F}(d)$ we define (forced by naturality)

$$\begin{aligned} \bar{f}(d)(c \xrightarrow{\varphi} d) &= \bar{f}(d)(c \xrightarrow{\mathrm{id}} c \xrightarrow{\varphi} d) \\ &= \mathcal{M}(\varphi)(\underbrace{\bar{f}(c)(c \xrightarrow{\mathrm{id}} c)}_{\in \mathcal{M}(c)}) \in \mathcal{M}(d) \\ &= \mathcal{M}(\varphi)(m_0) . \end{aligned}$$

One easily checks that $\bar{f}$ defined as above is a natural transformation.

$\boxtimes$

Example: $C_n^{\mathrm{sing}}(\_;R) : \underset{\overset{\shortparallel}{\mathcal{C}}}{\mathrm{Top}} \longrightarrow R\text{-}\mathcal{M}od$

is a free $R\mathrm{Top}$-module: $C_n^{\mathrm{sing}}(-;R) = R[\mathrm{mor}_{\mathrm{Top}}(\Delta_n, -)]$

$C_*^{\mathrm{sing}}(\_;R)$ is a free $R\mathcal{C}$-chain complex.

The following theorem has essentially the same proof as for R-modules:

↻

Theorem: Let F_* be a positive, projective $R\mathcal{C}$-chain complex. Let G_* be a positive $R\mathcal{C}$-chain complex such that

$$H_n(G_*(c)) = 0 \quad \text{for all } c \in ob(\mathcal{C}), \quad n > 0.$$

There is a bijection

$$[F_*, G_*] \xrightarrow{\sim} hom_{R\mathcal{C}}(H_0 \circ F_*, H_0 \circ G_*).$$

Theorem (acyclic model theorem):

Let F_* be a positive free $R\mathcal{C}$-chain complex. Let $\mathcal{M} \in ob(\mathcal{C})$ be the set of models of all F_n.

Let G_* be an $R\mathcal{C}$-chain complex s.th.

$$H_n(G_*(c)) = 0 \quad \text{for all } c \in ob(\mathcal{C}), \quad n > 0.$$

Then

$$[F_*, G_*] \xrightarrow{\cong} hom_{R\mathcal{C}}(H_0 \circ F_*, H_0 \circ G_*).$$

Proof of EZ chain homotopy equivalence:

16.11.

Let $\mathcal{C} = Top \times Top$.

Consider the two $R\mathcal{C}$-chain complexes

$$F_* = C_*^{sing}(\_ \times \_ ; R),$$

$$G_* = C_*^{sing}(\_ ; R) \underset{R}{\otimes} C_*^{sing}(\_ ; R).$$

We have

$$F_n = R[mor_{Top}(\Delta_n, \_ \times \_)] \cong R[mor_{\mathcal{C}}(\Delta_n \times \Delta_n, \_ \times \_)]$$

since $mor_{Top}(\Delta_n, \_ \times \_)$ and $mor_{\mathcal{C}}(\Delta_n \times \Delta_n, \_ \times \_)$ are naturally isomorphic as functors from $\mathcal{C}$ to Sets.

Thus F_n is a free $R\mathcal{C}$-module with model $\Delta_n \times \Delta_n$.

↗

$$G_n = \bigoplus_{p+q=n} C_p^{sing}(-;R)\otimes_R C_q^{sing}(-;R)$$

$$= \bigoplus_{p+q=n} R[\mathrm{mor}_{Top}(\Delta_p,-)]\otimes_R R[\mathrm{mor}_{Top}(\Delta_q,-)]$$

$$\cong \bigoplus_{p+q=n} R[\underbrace{\mathrm{mor}_{Top}(\Delta_p,-)\times \mathrm{mor}_{Top}(\Delta_q,-)}_{\cong\ \mathrm{mor}_{\mathcal{C}}(\Delta_p\times\Delta_q,\,\_\times\_)}]\ .$$

Thus G_n is free with models $\Delta_p\times\Delta_q$, $p+q=n$.

Since $\Delta_p\times\Delta_q\simeq *$ for all p,q, the chain complexes

$$F_*(\Delta_p\times\Delta_q) \quad \text{and} \quad G_*(\Delta_p\times\Delta_q)$$

have vanishing homology in degrees >0.

So the acyclic model theorem implies the EZ chain homotopy result. $\square$

—

7. Cohomology and products

7.1 Cohomology theories

A cohomology theory (h^*,δ^*) is a family of contravariant functors

$$h^n : Top^{(2)} \longrightarrow R\text{-modules},$$

(R being a commutative ring)

together with boundary homomorphisms δ^n, i.e.

natural transformations $h^n\circ J \xrightarrow{\delta^n} h^{n+1}$, where $J: \underset{(X,A)\,\mapsto\,(A,\emptyset)}{Top^{(2)}\to Top^{(2)}}$,

such that the obvious analogues of

homotopy invariance, excision, LES hold,

where all arrows are reversed.

↺

Def.: The n-th singular cohomology (with coefficients in R) is defined as

$$H^n(X,A;R) = H^n(\hom_R(C_*^{sing}(X,A;R),R)) .$$

Rem.:
$$\begin{aligned}\hom_R(C_i^{sing}(X,A;R),R) &\cong \hom_R(R\otimes C_i^{sing}(X,A;\mathbb{Z}),R)\\ &\cong \hom_{\mathbb{Z}}(C_i^{sing}(X,A),R)\end{aligned}$$

If $A=\emptyset$ then it is also isomorphic to $map(S_i(X),R)$.

Proof of the axioms:

1) LES:
$$\begin{tikzcd}0 \arrow[r] & C_*^{sing}(A) \arrow[r] & C_*^{sing}(X) \arrow[r, "p_*"] & C_*^{sing}(X,A) \arrow[r] \arrow[l, bend right, dashed] & 0\end{tikzcd}$$

free $\mathbb{Z}$-module with $\mathbb{Z}$-basis
$\{p_*(\sigma) \mid \sigma\in S_i(X),\ im(\sigma)\not\subseteq A\}$

So the sequence splits in each degree. Hence

$$0 \to \hom(C_*^{sing}(X,A),R) \to \hom_{\mathbb{Z}}(C_*^{sing}(X),R) \to \hom_{\mathbb{Z}}(C_*^{sing}(A),R) \to 0$$

is also exact.

The associated LES is the desired LES in cohomology.

$$\dots \to H^n(X,A;R) \to H^n(X;R) \to H^n(A;R) \xrightarrow{\delta^n} H^{n+1}(X,A;R) \to \dots$$

2) Homotopy invariance:

$$\begin{aligned} f\simeq g \quad &\Rightarrow \quad C_*^{sing}(f) \simeq C_*^{sing}(g)\\ &\Rightarrow \quad \hom(C_*^{sing}(f), id_R) \simeq \hom(C_*^{sing}(g), id_R)\\ &\Rightarrow \quad H^n(f) = H^n(g)\end{aligned}$$

↗

3) Excision: $A \subset B \subset X$ with $\bar{A} \subset B^{\circ}$.

Consider <u>just for a PID</u> R

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\operatorname{Ext}^1_R(H_{n-1}(X,B;R),R) \arrow[r, "\cong ①"] \arrow[d] & \operatorname{Ext}^1_R(H_{n-1}(X\setminus A, B\setminus A;R),R) \arrow[d] \\
H^n(X,B;R) \arrow[r] \arrow[d] & H^n(X\setminus A, B\setminus A) \arrow[d] \\
\hom_R(H_n(X,B;R),R) \arrow[r, "\cong ②"] \arrow[d] & \hom_R(H_n(X\setminus A, B\setminus A;R),R) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The columns are exact by the universal coefficient theorem, the arrows ① and ② are iso's by excision in homology.

5-lemma $\Longrightarrow$ claim for PID's.

4) Dimension axiom:
$$H^n(*;R) \cong \begin{cases} R & \text{if } n=0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

This follows either from explicit computation or the universal coefficient theorem.

5) Additivity axiom:

$$H^n\Big(\coprod_{i\in I} X_i\,;R\Big) \xrightarrow{\cong} \prod_{i\in I} H^n(X_i;R)$$

follows from

$$\begin{aligned}\hom_{\mathbb{Z}}\Big(C_*^{sing}\big(\coprod X_i\big),R\Big) &\cong \hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} C_*^{sing}(X_i), R\Big) \\ &\cong \prod_{i\in I} \hom_{\mathbb{Z}}(C_*^{sing}(X_i),R)\end{aligned}$$

⊠

Let us record for future reference:

<u>Theorem</u> (Universal coefficient theorem for singular cohomology):

Let R be a PID.

$$0 \longrightarrow \operatorname{Ext}^1_R(H_{n-1}(X,A;R),R) \longrightarrow H^n(X,A;R) \longrightarrow \hom_R(H_n(X,A;R),R) \longrightarrow 0$$

is a natural exact sequence which splits.

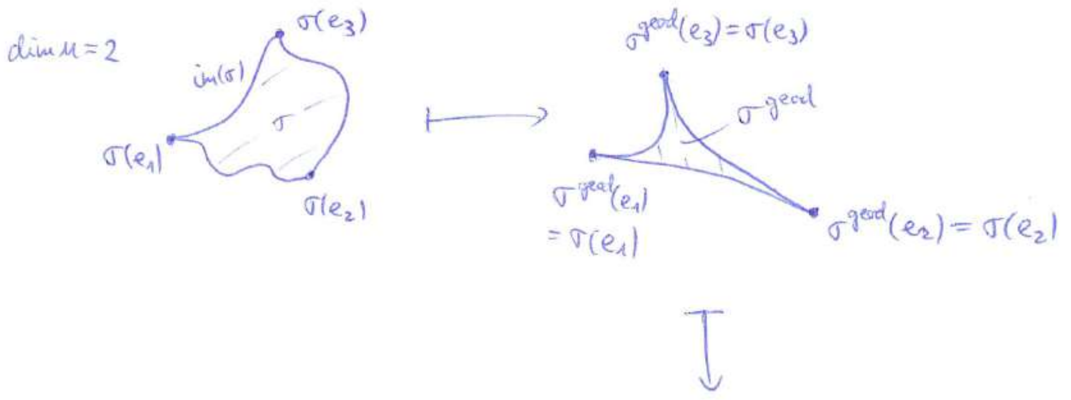
<u>Proof</u>: Direct consequence of the corresponding statement for cochain complexes.

What is it good for?

1) Very often, geometry provides natural cocycles (instead of cycles).

One example: Let M be a non-positive curved Riemannian manifold, M simply connected, e.g. $M = \mathbb{R}^n$.

<u>volume cocycle</u> $\qquad \varphi : S_{\dim(M)}(M) \longrightarrow \mathbb{R}$



$$\operatorname{Vol}(\sigma^{geod}) = \int_{\Delta_{\dim M}} (\sigma^{geod})^* \, dvol_M$$

↗

2) Close relation to analysis on manifolds :

M smooth manifold

$$\Omega^*(M) \xrightarrow{\quad A_* \quad} \hom_{\mathbb{Z}}\big(C_*^{\overset{\text{smooth}}{\text{sing}}}(M), \mathbb{R}\big) \xleftarrow[\text{(not trivial)}]{\cong} \hom_{\mathbb{Z}}\big(C_*^{\text{sing}}(M), \mathbb{R}\big)$$

cochain complex of differential forms

$$\Omega^p(M) \ni \omega \longmapsto \left(\begin{array}{rcl} C_p^{\text{sing}}(M) & \longrightarrow & \mathbb{R} \\ (\sigma : \Delta_p \to M) & \longmapsto & \int_{\Delta_p} \sigma^* \omega \end{array} \right)$$

Stokes thus implies that A_* is a cochain map :

$$A^{p+1}(d\omega)(\rho : \Delta_{p+1} \to M) = \int_{\Delta_{p+1}} \rho^*(d\omega) = \int_{\Delta_{p+1}} d\,\rho^*\omega$$

$$\overset{\text{Stokes}}{=} \int_{\partial\Delta_{p+1}} \rho^*\omega = A^p(\omega)(d\rho) = \delta(A^p(\omega)(\rho))$$

Theorem (de Rham):

A^* induces an isomorphism in cohomology.

3) H^* has a product structure :

in degree zero : A 0-cocycle is a map $\varphi : S_0(X) \to R$.

You can multiply these functions pointwise.

This is part of a product

$$H^p(X;R) \times H^q(X;R) \longrightarrow H^{p+q}(X;R).$$

↺

7.2 Multiplicative structures

Def.: Let h^* be a cohomology theory with coefficients in R. A multiplicative structure assigns to any space X and subspaces $A, B \subset X$ a family of R-bilinear maps

$$\cup : h^p(X,A) \times h^q(X,B) \longrightarrow h^{p+q}((X,A) \times (X,B)) \qquad (*)$$

and to every space Y an element $1_Y \in h^0(Y)$ such that the following holds:

1) $\cup$ is natural with respect to maps of triples
$$(X; A, B) \longrightarrow (X'; A', B')$$

2) $\cup$ is (graded) commutative:
For $u \in h^p(X,A)$ and $v \in h^q(X,B)$ we have
$$u \cup v = (-1)^{pq} v \cup u .$$

3) $\cup$ is associative.

4) For $u \in h^p(X,A)$ we have $1_X \cup u = u \cup 1_X = u$.

5) Let $j : A \hookrightarrow X$ be an inclusion of a subspace.
Let $u \in h^p(A)$ and $v \in h^q(X)$. Then:
$$\delta^p(u) \cup v = \delta^{p+q}(u \cup h^q(j)(v))$$

Rem.: Most often, $(*)$ is not defined for all triples $(X; A, B)$ but only for a restricted class (later: for those s.th. (A,B) is excisive). If all the properties hold whenever applicable, we still speak of a multiplicative structure.

↗

7.3 (Graded) algebras

17.11.

Def.: An R-module A together with an R-bilinear map

$$m: A \times A \longrightarrow A$$

(equivalently, an R-homomorphism $A \otimes_R A \to A$)

is an R-algebra if $(A, +, m)$ is a ring.

A graded R-algebra is an R-algebra $A = \bigoplus_{p \in \mathbb{Z}} A^p$ s.th. for $u \in A^p$ and $v \in A^q$ the product $m(u,v)$ is in A^{p+q}.

Elements in A^p for some p are called homogeneous.

We call a graded R-algebra $A = \oplus A^p$ graded commutative if $m(u,v) = (-1)^{pq} m(v,u)$ for $u \in A^p$ and $v \in A^q$ holds.

The degree p of a homogenous element $u \in A^p$ is denoted by $|a|$ or $\deg(a)$.

Rem.: If h^* is a cohomology theory theory (in R) with a multiplicative structure $\cup$, then

$$h^*(X) = \bigoplus_{p \in \mathbb{Z}} h^p(X)$$

is a graded R-algebra which is graded commutative.

Examples: 1) The polynomial ring $R[X_1, \dots, X_n]$ is an R-algebra.

Now suppose $X_1, \dots, X_n$ are assigned degrees $\deg X_i \in \mathbb{N}$. Then

$$V = R[X_1, \dots, X_n]$$

is a graded R-module

$$V = \bigoplus_{p \in \mathbb{N}_0} V^p$$

where V^p is generated by monomials

↺

$X_1^{j_1} \cdots X_n^{j_n}$ with $p = j_1 \deg(X_1) + \ldots + j_n \deg(X_n)$.

With the usual multiplication $V = R[X_1, \ldots, X_n]$ becomes a graded R-algebra.

It is graded commutative if $\deg(X_i)$ is even for all $i \in \{1, \ldots, n\}$ or $2 = 0$ in R.

(It is, of course, commutative in the usual sense!)

2) The <u>exterior algebra</u> $\Lambda_R[X_1, \ldots, X_n]$ generated by $X_1, \ldots, X_n$ with odd degrees (preassigned) $\deg(X_i) \in \mathbb{N}$ is the free R-module generated by the finite (formal) products

$$X_{i_1} \cdots X_{i_k} \quad , \quad i_1 < i_2 < \ldots < i_k \, .$$

It is graded by

$$V^p = \text{free } R\text{-module generated by } X_{i_1} \cdots X_{i_k}, \; i_1 \deg(X_{i_1}) + \ldots + i_k \deg(X_{i_k}) = p \, .$$

The multiplication is defined by the rules

$$X_i X_j = -X_j X_i \quad \text{and} \quad X_i X_i = 0 \, .$$

It is graded commutative.

<u>Def.:</u> If $A^* = \bigoplus A^p$ and $B^* = \bigoplus B^p$ are graded R-algebras, then $A^* \underset{R}{\otimes} B^* = \bigoplus_{n \in \mathbb{Z}} \left(\bigoplus_{p+q=n} A^p \underset{R}{\otimes} B^q \right)$ becomes a graded R-algebra via

$$(a \otimes b)(a' \otimes b') := (-1)^{|a'||b|} aa' \otimes bb'$$

for $a \in A^p$, $a' \in A^{p'}$, $b \in B^q$, $b' \in B^{q'}$.

If A^* and B^* are graded commutative, then $A^* \underset{R}{\otimes} B^*$ is graded commutative.

↗

Example: $\Lambda_R[X_1,\dots,X_n]\otimes_R \Lambda_R[Y_1,\dots,Y_m]\cong \Lambda_R[X_1,\dots,X_n,Y_1,\dots,Y_m]$

7.4 Multiplicative structures on singular cohomology.

Write $C_* = C_*^{sing}$.

Consider the natural cochain map

$$\begin{tikzcd}
\hom_{\mathbb{Z}}(C_*(X),R)\otimes_R \hom_{\mathbb{Z}}(C_*(Y),R) \arrow[d] & f\otimes g \arrow[d, mapsto] \\
\hom_{\mathbb{Z}}(C_*(X)\otimes_{\mathbb{Z}} C_*(Y),\, R\otimes_{\mathbb{Z}} R) & (x\otimes y \longmapsto f(x)\otimes g(x)).
\end{tikzcd}$$

If we compose this map with $\hom_{\mathbb{Z}}(AW_*,\, R\otimes_{\mathbb{Z}} R \underset{\text{multiplication}}{\longrightarrow} R)$,

then we obtain a cochain map

$$\begin{tikzcd}
\hom_{\mathbb{Z}}(C_*(X),R)\otimes_R \hom_{\mathbb{Z}}(C_*(Y),R) \arrow[d] \\
\hom_{\mathbb{Z}}(C_*(X\times Y),R)
\end{tikzcd}$$

Taking cohomology, we obtain a map

$$\begin{tikzcd}
\underbrace{H^p(\hom_{\mathbb{Z}}(C_*(X),R))}_{H^p(X,R)}\otimes_R \underbrace{H^q(\hom_{\mathbb{Z}}(C_*(Y),R))}_{H^q(Y,R)} \arrow[d] & {[f]\otimes[g]} \arrow[d, mapsto] \\
H^{p+q}(\hom_{\mathbb{Z}}(C_*(X),R)\otimes_R \hom_{\mathbb{Z}}(C_*(Y),R)) \arrow[d] & {[f\otimes g]} \\
H^{p+q}(X\times Y,R) &
\end{tikzcd}$$

which is called the <u>cross product</u> (in cohomology), and denoted by $\times$.

Def.: We define the cup_product

$$\cup : H^p(X, R) \underset{R}{\otimes} H^q(X; R) \longrightarrow H^{p+q}(X; R)$$

by the composition

$$H^p(X,R) \underset{R}{\otimes} H^q(X,R) \xrightarrow{\times} H^{p+q}(X \times X, R) \xrightarrow{H^{p+q}(\Delta)} H^{p+q}(X,R) \quad ,$$

where $\Delta : X \to X \times X$ is the diagonal map $\Delta(x) = (x,x)$.

Explicitly, we have for a p-cocycle φ and a q-cocycle ψ:

$$[\varphi] \cup [\psi] = \left[\sigma \longmapsto \varphi\left(\sigma_{|[e_1, \dots, e_{p+1}]}\right) \cdot \psi\left(\sigma|_{[e_{p+1}, \dots, e_{p+q+1}]}\right) \right] .$$

Suppose $\varphi \in \hom_{\mathbb{Z}}(C_p(X,A), R) = \left\{ S_p(X) \xrightarrow{\tau} R \mid \tau \text{ vanishes on } S_p(A) \subset S_p(X) \right\}$

and $\psi \in \hom_{\mathbb{Z}}(C_q(X,B), R) = \left\{ S_q(X) \xrightarrow{\nu} R \mid \nu \text{ vanishes on } S_q(B) \subset S_q(X) \right\}$

are given. From the explicit formula for $\cup$ we see that

$$\begin{tikzcd} \varphi \cup \psi \in \hom_{\mathbb{Z}}\left(C_*(X) / C_*(A) + C_*(B) , R \right) \\ \hom_{\mathbb{Z}}(C_*(X, A \cup B), R) \arrow[u, "\text{— H^*-iso provided (A,B) is excisive (for $\mathbb{Z}$-coefficients)}"'] \end{tikzcd}$$

Therefore, if (A,B) is excisive, we obtain

$$\cup : H^p(X, A; R) \underset{R}{\otimes} H^q(X, B; R) \longrightarrow H^{p+q}(X, A \cup B; R) .$$

Theorem: The cup-product gives a multiplicative structure on singular cohomology.

Proof: We only treat the absolute case.

$1_X \in H^0(X; R)$ is represented by the constant cocycle $\begin{array}{c} S^0(X) \longrightarrow R \\ \sigma \longmapsto 1 \end{array}$.

↗

Associativity and compatibility with the boundary homomorphism are easy to verify.

The graded commutativity is more involved (and does not hold on the cochain level!). The map

$$p_* : C_*(X) \longrightarrow C_*(X)$$
$$\sigma \longmapsto (-1)^{\frac{n(n+1)}{2}} \cdot \underbrace{\sigma|_{[e_{n+1},\dots,e_1]}}_{\substack{= \\ \left(\begin{smallmatrix}\Delta_n \to \Delta_n \xrightarrow{\sigma} X \\ e_i \mapsto e_{n+2-i}\end{smallmatrix}\right)}}$$

is a natural chain map.

We can regard it as a chain map of $\mathbb{Z}Top$-modules

$C_*(\_) \xrightarrow{p_*} C_*(\_)$.

Since $p_0 = id$, the acyclic model theorem implies that $p_* \simeq id_{C_*(\_)}$.

Further, p_* satisfies (write $p^*\varphi$ for the induced map on cocycles by p_*)

$$p^*\varphi \cup p^*\psi = (-1)^{|\varphi|\cdot|\psi|} \cdot p^*(\psi \cup \varphi) .$$

⌜ We check this up to sign:

$$(p^*\varphi \cup p^*\psi)(\sigma) = (p^*\varphi)(\sigma|_{[e_1,\dots,e_{p+1}]}) \cdot (p^*\psi)(\sigma|_{[e_{p+1},\dots,e_{p+q+1}]})$$
$$= \pm\, \varphi(\sigma|_{[e_{p+1},\dots,e_1]}) \cdot \psi(\sigma|_{[e_{p+q+1},\dots,e_{p+1}]})$$

$$(p^*(\psi \cup \varphi))(\sigma) = (\psi \cup \varphi)(\sigma|_{[e_{p+q+1},\dots,e_1]})$$
$$= \psi(\sigma|_{[e_{p+q+1},\dots,e_{p+1}]}) \cdot \varphi(\sigma|_{[e_{p+1},\dots,e_1]})$$ ⌟

Since $p^* \simeq id$ in the cochain complex, we obtain

$$[\varphi] \cup [\psi] = (-1)^{|\varphi|\cdot|\psi|} [\psi] \cup [\varphi] .$$ ▨

Proposition: The crossproduct in cohomology is a homomorphism of graded R-algebras

$$H^*(X;R) \underset{R}{\otimes} H^*(Y;R) \xrightarrow{\times} H^*(X\times Y;R) .$$

23.11.

Remark: One can relate cross and cup product as follows: let $pr_X : X\times Y \to X$ and $pr_Y : X\times Y \to Y$ be the projections. Consider the diagram

$$\begin{tikzcd}
H^*(X;R) \underset{R}{\otimes} H^*(Y;R) \arrow[r, "\times"] \arrow[d, "H^*(pr_X) \underset{R}{\otimes} H^*(pr_Y)"] \arrow[dr, phantom, "\circlearrowleft \text{ by naturality of } \times"] & H^*(X\times Y;R) \arrow[d, "H^*(pr_X \times pr_Y)"] \arrow[dd, bend left=70, "id"] \\
H^*(X\times Y;R) \underset{R}{\otimes} H^*(X\times Y;R) \arrow[r, "\times"] \arrow[dr, bend right, "\cup"'] \arrow[dr, phantom, "\circlearrowleft \text{ by definition of } \cup"] & H^*((X\times Y)\times(X\times Y);R) \arrow[d, "H^*(\Delta)"] \\
 & H^*(X\times Y;R)
\end{tikzcd}$$

where $\Delta : X\times Y \to (X\times Y)\times(X\times Y)$ is the diagonal.

Hence, we get

$$H^p(pr_X)(\varphi) \cup H^q(pr_Y)(\psi) = \varphi \times \psi$$

for $\varphi \in H^p(X;R)$ and $\psi \in H^q(Y;R)$.

The analogous relation holds in the relative case, i.e. for $\varphi \in H^p(X,A;R)$ and $\psi \in H^q(Y,B;R)$.

↗

Proof of proposition:

Let us denote for a moment the cross product by m.

To show: $\quad m((x_1 \otimes y_1)(x_2 \otimes y_2)) = m(x_1 \otimes y_1) \cup m(x_2 \otimes y_2)$.

left hand side: $\quad (x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{|y_1||x_2|} (x_1 \cup x_2) \otimes (y_1 \cup y_2)$

$\quad m(\quad - \quad) = (-1)^{|y_1||x_2|} (x_1 \cup x_2) \times (y_1 \cup y_2)$

right hand side:

$$
\begin{aligned}
(x_1 \times y_1) \cup (x_2 \times y_2) &= (pr_X^*(x_1) \cup pr_Y^*(y_1)) \cup (pr_X^*(x_2) \times pr_Y^*(y_2)) \\
&= (-1)^{|y_1||x_2|} pr_X^*(x_1) \cup pr_X^*(x_2) \cup pr_Y^*(y_1) \cup pr_Y^*(y_2) \\
&= (-1)^{|y_1||x_2|} pr_X^*(x_1 \cup x_2) \cup pr_Y^*(y_1 \cup y_2) \\
&= (-1)^{|y_1||x_2|} (x_1 \cup x_2) \times (y_1 \cup y_2) .
\end{aligned}
$$

▨

Theorem (Künneth theorem for cohomology of spaces):

Let R be a PID. Let (X,A) and (Y,B) be pairs of spaces s.th. $(A \times Y, X \times B)$ is excisive.

Let $H_p(X,A;R)$ be finitely generated for every p.

Then there is a short exact sequence (which splits):

$$
0 \longrightarrow \bigoplus_{p+q=n} H^p(X,A;R) \otimes_R H^q(Y,B;R) \xrightarrow{\times} H^n((X,A)\times(Y,B);R) \longrightarrow \bigoplus_{p+q=n+1} \operatorname{Tor}_1^R(H^p(X,A;R), H^q(Y,B;R)) \longrightarrow 0
$$

Proof: Similar to Künneth for homology.

We just consider the case where R is a field, so all (co-)homology modules are free.

↻

$$\begin{tikzcd}
\bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(Y;R) \arrow[r, "\times"] \arrow[d, "\cong \text{ by UCT}"] & H^*(X\times Y;R) \arrow[ddd, "\cong \text{ by UCT}"] \\
\bigoplus_{p+q=n} \hom_R(H_p(X;R),R) \otimes_R \hom_R(H_q(Y;R),R) \arrow[d, "\cong \text{ because } H_*(X;R) \text{ are fin. gen.}", "\varphi\otimes\psi \mapsto (x\otimes y \mapsto \varphi(x)\cdot\psi(y))"'] \arrow[dr, phantom, "\circlearrowright"] & \\
\bigoplus_{p+q=n} \hom_R(H_p(X;R) \otimes_R H_q(Y;R), R) \arrow[d, "\cong"] & \\
\hom_R\Big(\bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(Y;R), R\Big) & \hom_R(H_n(X\times Y;R),R) \arrow[l, "\cong"', "\text{induced by homological cross product}"]
\end{tikzcd}$$

The lower horizontal map is an isomorphism by the homological Künneth theorem. The claim in this special case follows from the commutativity of the diagram. (!) ☒

7.5 Examples of computations of cup products

Example: S^1

$$H^*(S^1) = H^0(S^1) \oplus H^1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Denote a generator of $\begin{cases} H^0(S^1) \\ H^1(S^1) \end{cases}$ by $\begin{cases} 1_{S^1} \\ \alpha \end{cases}$.

UCT: $H^1(S^1) \xrightarrow{\cong} \hom(\underset{\underset{\mathbb{Z}}{\parallel}}{H_1(S^1)}, \mathbb{Z})$ ($H^1(-;\mathbb{Z})$ always torsion-free)

The cup product is determined by this!

$$\left.\begin{array}{l} \alpha \cup 1_{S^1} = 1_{S^1} \cup \alpha = \alpha \\ \alpha \cup \alpha \in H^2(S^1) = 0 \end{array}\right\} \Rightarrow H^*(S^1) \cong \Lambda_{\mathbb{Z}}[\alpha], \quad |\alpha| = 1.$$

Example: $T^2 = S^1 \times S^1$

Cohomological Künneth theorem:

$$\begin{tikzcd} H^*(S^1) \otimes_{\mathbb{Z}} H^*(S^1) \arrow[r, "\times", "\cong"'] & H^*(S^1 \times S^1) \end{tikzcd}$$

$$\underset{\Lambda_{\mathbb{Z}}[\alpha_1]}{\overset{H^*(S^1)}{\parallel}} \quad \underset{\Lambda_{\mathbb{Z}}[\alpha_2]}{\overset{H^*(S^1)}{\parallel}}$$

$$\Rightarrow \quad H^*(S^1 \times S^1) \cong \Lambda_{\mathbb{Z}}[\alpha_1, \alpha_2]$$

H^0 : $\mathbb{Z} \cdot 1_{T^2}$

H^1 : $\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$

H^2 : $\mathbb{Z}\alpha_1\alpha_2$

Inductively, we get $H^*(T^n) \cong \Lambda_{\mathbb{Z}}[\alpha_1, \dots, \alpha_n]$

↑

has rank $\binom{n}{p}$

in degree p

↺

Let's give S^1 the minimal structure.

Let's denote by e_i the 1-cell in the i-th factor of T^2, $i \in \{1,2\}$.

Of course, the product CW-structure is the "usual" for T^2.

"double meanings":

e_i 1-cell $\longleftrightarrow$ e_i generator of $H_1(S^1)$

$e_1 \times e_2$ 2-cell in T^2 $\longleftrightarrow$ cross product $e_1 \times e_2 \in H_2(T^2)$

UCT: $$H^1(S^1) \xrightarrow{\cong} \hom(\underset{\substack{\cup \\ e_i}}{H_1(S^1)}, \mathbb{Z})$$

α_i : dual basis

$(\alpha_i(e_i) = 1)$

homological cross product

$$\rightarrow (\alpha_1 \times \alpha_2)(e_1 \times e_2) = 1$$

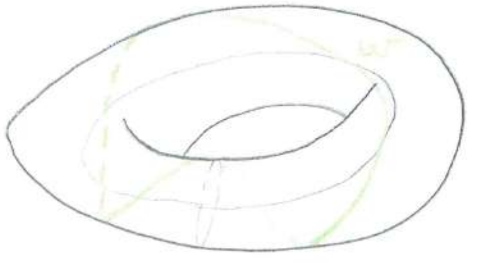
$\alpha_i \in H^1(S^1)$

$$(\beta_1 \cup \beta_2)(\overset{\text{2-cell}}{\underset{\text{gen. of } H_2(T^2)}{e_1 \times e_2}}) = \pm 1$$

$H_1(T^2) = \mathbb{Z}[e_1, e_2]$; dual basis of $\underset{\substack{\cong \\ \hom_{\mathbb{Z}}(H_1(T^2), \mathbb{Z})}}{H^1(T^2)}$: β_1, β_2 with $\beta_1(e_1) = 1$, $\beta_2(e_1) = 0$, $\beta_1(e_2) = 0$, $\beta_2(e_2) = 1$

$$\beta_1 = H^1(pr_\ell)(\alpha) \overset{!}{=} \alpha \times 1_{S^1}, \quad \beta_2 = H^1(pr_r)(\alpha) \overset{!}{=} 1_{S^1} \times \alpha$$

since e.g. $\alpha \times 1_{S^1} = H^1(pr_\ell)(\alpha) \cup H^0(pr_r)(1_{S^1}) = H^1(pr_\ell)(\alpha) \cup 1_{T^2} = \beta_1 \cup 1_{T^2} = \beta_1$.



$\beta_1(w) = \#$ w winds around equator e_1

$\beta_2(w) = \#$ w —"— meridian e_2

$$\underset{\substack{\cup \\ w}}{\pi_1(T^2)} \cong H_1(T^2) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$$

Example: Find a space X with $H^*(X) \cong \Lambda_{\mathbb{Z}}[\alpha_1,\alpha_2,\alpha_3]/_{(\alpha_2\alpha_3)}$, $|\alpha_i| = 1$.

24.11.

$H^0 \cong \mathbb{Z}\, 1_X$

$H^1 \cong \mathbb{Z}[\{\alpha_1,\alpha_2,\alpha_3\}]$

$H^2 \cong \mathbb{Z}[\{\alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_1\alpha_3\}]/_{\mathbb{Z}[\{\alpha_2\alpha_3\}]} \cong \mathbb{Z}[\{\alpha_1\alpha_2, \alpha_1\alpha_3\}]$

$H^3 \cong 0$

Idea: Construct X as a subcomplex $j : X \hookrightarrow T^3$.

Note that $H^*(j) : H^*(T^3) \longrightarrow H^*(X)$ is surjective:

$$\begin{tikzcd} C^{cw}_*(j): & C^{cw}_*(X) \arrow[r, hook, "inj."'] \arrow[d, squiggly] & C^{cw}_*(T^3) \\ C^*_{cw}(j): & \hom_{\mathbb{Z}}(C^{cw}_*(T^3), \mathbb{Z}) \arrow[r, two heads] & \hom_{\mathbb{Z}}(C^{cw}_*(X), \mathbb{Z}) \end{tikzcd}$$

Differentials are zero, hence

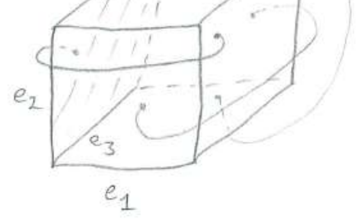
$$H^*(j) : H^*(T^3) \twoheadrightarrow H^*(X) \quad \text{surjective.}$$

So we have to find a subcomplex $j : X \hookrightarrow T^3$ s.th.

$$\ker(H^*(j)) = (\alpha_2 \cup \alpha_3),$$

where we identify $H^*(T^3) = \Lambda_{\mathbb{Z}}[\alpha_1,\alpha_2,\alpha_3]$ and α_i are the generators of $H^1(T^3)$.

T^3 :



To get X, we delete the 2-cell $e_2 \times e_3$.

To get a CW-complex, we also have to delete the 3-cell $e_1 \times e_2 \times e_3$.

Proposition: Let X and Y be path-connected spaces.

Consider $X \vee Y$:

$$\begin{tikzcd}
pt \arrow[r, hook] \arrow[d, "\text{closed neighborhood deformation retract}"'] & Y \arrow[d, "j_Y"] \\
X \arrow[r, "j_X"] & X \vee Y
\end{tikzcd}$$

We obtain from MV in cohomology:

$$H^i(X \vee Y) \cong \begin{cases} \mathbb{Z} , & \text{if } i=0 , \\ H^i(X) \oplus H^i(Y) , & \text{if } i>0 . \end{cases}$$

Let $\alpha \in H^i(X)$ and $\beta \in H^j(Y)$ for $i,j > 0$.

Let $p_X : X \vee Y \to X$ and $p_Y : X \vee Y \to Y$ be the "projections".

We have $\begin{cases} p_X \circ j_X = id_X , \\ p_Y \circ j_Y = id_Y . \end{cases}$

Claim: $H^i(p_X)(\alpha) \cup H^j(p_Y)(\beta) = 0$

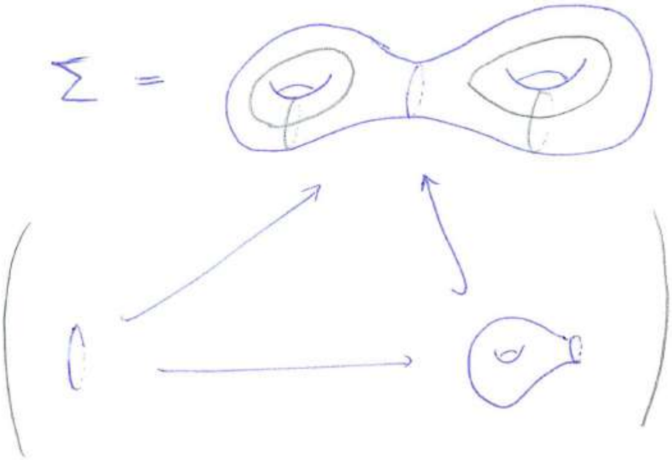
$$\begin{aligned} H^{i+j}(j_X)\big(H^i(p_X)(\alpha) \cup H^j(p_Y)(\beta)\big) &= H^i(j_X)\big(H^i(p_X)(\alpha)\big) \cup H^j(j_X)\big(H^j(p_Y)(\beta)\big) \\ &= \alpha \cup H^j(p_Y \circ j_X)(\beta) \\ &= \alpha \cup 0 \\ &= 0 . \end{aligned}$$

Symmetrically, we get that $H^{i+j}(j_Y)\big(H^i(p_X)(\alpha) \cup H^j(p_Y)(\beta)\big)$.

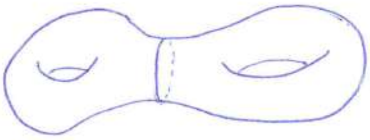
Together, this implies the claim.

Example:

$\Sigma =$

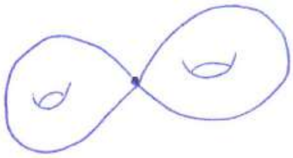


Homology groups of Σ and X are all free $\mathbb{Z}$-modules.



$\Big\downarrow$ pr (shrinking the middle circle)

UCT $\rightsquigarrow$ $\begin{cases} H^* = \hom_{\mathbb{Z}}(H_*, \mathbb{Z}) \\ H^*(pr) = \hom_{\mathbb{Z}}(H_*(pr), \mathbb{Z}) \end{cases}$



$T^2 \vee T^2 =: X$

$H_1(pr)$ (thus $H^1(pr)$) is an isomorphism.

One sees geometrically that $H_1(pr)$ maps generators to generators.

What about $H_2(pr)$?

$H_2(\Sigma) \cong \mathbb{Z}$, $H_2(X) \cong H_2(T^2) \oplus H_2(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$

$H_2(pr)$ corresponds to the diagonal $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $x \mapsto (x, x)$ under these isomorphisms:

To this end, consider

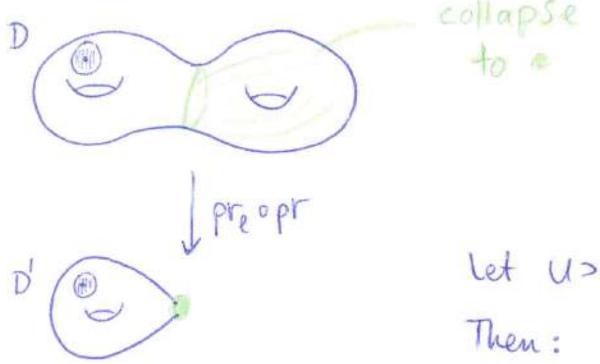
projection to left torus

$$\Sigma \xrightarrow{pr} X \xrightarrow{pr_\ell} T^2$$

and similarly the projection pr_r to the right torus.

⤴

To show: $H_2(pr_e \circ pr)$ and $H_2(pr_r \circ pr)$ are isomorphisms.



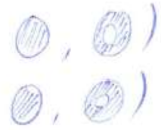
Let $A = \Sigma \setminus \mathring{D}$

$A' = T^2 \setminus \mathring{D}'$.

Let $U \supset A$ be a slightly bigger open set in Σ.

Then:

$$pr_e \circ pr \left(\cong \begin{matrix} (\Sigma \setminus A, U \setminus A) \cong (\quad , \quad) \\ (T^2 \setminus A', U' \setminus A') \cong (\quad , \quad) \end{matrix} \right.$$



$$\begin{tikzcd}
H^2(\Sigma) & H^2(\Sigma, U) \arrow[l, two heads, "①"'] \arrow[r, "\cong", "\text{excision}"'] & H^2(\Sigma \setminus A, U \setminus A) \cong H^2(D^2, S^1) \cong \mathbb{Z} \\
H^2(T^2) \arrow[u, "\cong", "H^2(pr_e \circ pr)"'] & H^2(T^2, U') \arrow[l, two heads, "②"'] \arrow[u] \arrow[r, "\cong", "\text{excision}"'] & H^2(T^2 \setminus A', U' \setminus A') \arrow[u, "③", "\cong"']
\end{tikzcd}$$

① and ② are surjective because U' and U are homotopy equivalent to 1-dim. CW complexes, hence their 2nd cohomology vanishes.

③ is an isomorphism since it is induced by a homeomorphism of pairs of spaces.

$H^*(\Sigma)$: $\quad H^0 = \mathbb{Z} \cdot 1_\Sigma$

$H^1 = \mathbb{Z} \cdot \alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_2$

$H^2 = \mathbb{Z} \cdot \omega$

Cup products in $H^*(\Sigma)$:

$\alpha_1 \cup \alpha_2 = \omega$

$\beta_1 \cup \beta_2 = \omega$

$\alpha_i \cup \beta_j = 0$

follows immediately from the previous proposition

$$\begin{tikzcd}
 & \mathbb{Z} & 1 \\
H^1(\Sigma) \otimes_{\mathbb{Z}} H^1(\Sigma) \arrow[r, "\cup"] & H^2(\Sigma) \arrow[u, phantom, "\cong"] & \\
H^1(X) \otimes_{\mathbb{Z}} H^1(X) \arrow[u, "H^1(pr) \otimes H^1(pr)"] \arrow[r, "\cup"] & H^2(X) \arrow[u, "\text{``+''}"'] & \\
 & \mathbb{Z} \oplus \mathbb{Z} \arrow[u, phantom, "\cong"] & (1,0) \arrow[uuu, mapsto] \\
H^1(T^2) \otimes_{\mathbb{Z}} H^1(T^2) \arrow[uu, "\text{induced by } pr_e"] \arrow[r, "\cup"] & H^2(T^2) \arrow[uu, bend right, "\Rightarrow"'] & \\
\alpha_1 \quad \alpha_2 \arrow[r, mapsto] & \text{generator} \arrow[uur, bend right] &
\end{tikzcd}$$

30.11

Theorem: $H^*(\mathbb{R}P^n;\mathbb{F}_2) = \mathbb{F}_2[\alpha]/_{(\alpha^{n+1})}$, $|\alpha| = 1$.

Proof: Write
$$\begin{cases} H^*(\_) = H^*(\_;\mathbb{F}_2) , \\ P^n = \mathbb{R}P^n , \\ \otimes = \otimes_{\mathbb{F}_2} . \end{cases}$$

Reminder on (co-)homology of P^n:

P^n has a CW-structure with exactly one i-cell for every $i \in \{0,\dots,n\}$ and the i-Skeleton of P^n is P^i. The attaching map for the i-cell in P^n is

$$S^{i-1} \xrightarrow[\substack{\uparrow \\ \text{2-fold} \\ \text{covering}}]{} P^{i-1} \longrightarrow P^{i-1}/_{P^{i-2}} \cong S^{i-1} .$$

Thus the degree of the attaching map is zero mod 2.
Hence $C_*^{CW}(P^n;\mathbb{F}_2)$ looks like

$$0 \longrightarrow \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{F}_2 \longrightarrow 0 .$$

$$\Rightarrow \quad \hom_{\mathbb{F}_2}(C_*^{CW}(P^n;\mathbb{F}_2),\mathbb{F}_2) = \Big(0 \leftarrow \underset{n}{\mathbb{F}_2} \xleftarrow{0} \cdots \xleftarrow{0} \underset{0}{\mathbb{F}_2} \leftarrow 0 \Big)$$

$$\Rightarrow \quad H^i(P^n) \cong \begin{cases} \mathbb{F}_2 , & i \in \{0,\dots,n\} , \\ 0 , & \text{otherwise} . \end{cases}$$

Moreover, we see that
$$P^{n-1} \hookrightarrow P^n$$
is an isomorphism on H^i for $i \leq n-1$.
So it suffices to prove by induction that the cup product of the generator of $H^{n-1}(P^n)$ with the generator of $H^1(P^n)$ is the generator of $H^n(P^n)$.

↺

In fact, we will prove that the cup product of the generator of $H^i(P^n)$ with the generator of $H^{n-i}(P^n)$ is the generator of $H^n(P^n)$.

Write $j = n-i$, so $i+j=n$.

Embed $P^i \hookrightarrow P^n$ and $P^j \hookrightarrow P^n$.
(last j coordinates zero) (first i coordinates zero)

Then $P^i \cap P^j$ is a single point $p \in P^n$, namely $p = [\underbrace{0,\dots,0}_{i},1,\underbrace{0,\dots,0}_{j}]$.

Let $p \in \mathcal{U} = \{[x_0,\dots,x_n] \in P^n \mid x_i \neq 0\} \xleftarrow{\cong} \mathbb{R}^n$.

$$[y_1,\dots,y_{i-1},1,y_i,\dots,y_n] \longleftarrow\!\shortmid (y_1,\dots,y_n)$$

$$[x_0,\dots,x_n] \longmapsto \left(\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\right)$$

Consider the diagram:

$$\begin{tikzcd}
H^i(P^n) \otimes H^j(P^n) \arrow[r, "\cup"] & H^n(P^n) \\
H^i(P^n, P^n \setminus P^j) \otimes H^j(P^n, P^n \setminus P^i) \arrow[u, "\textcircled{a}"] \arrow[r, "\cup"] \arrow[d, "\textcircled{b}"', "\text{induced by chart homeomorphism}"] & H^n(P^n, P^n \setminus \{p\}) \arrow[u, "\textcircled{c}"'] \arrow[d, "\textcircled{d}"] \\
H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j) \otimes H^j(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i) \arrow[r, "\cup"] & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})
\end{tikzcd}$$

We will prove:

1) The lower horizontal is an isomorphism.

2) All vertical maps are isomorphisms.

Ad 1) Let $pr_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^i = \mathbb{R}^i \times \{0\}$ be projections.
$pr_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^j = \{0\} \times \mathbb{R}^j$

pr_1 and pr_2 yield homotopy equivalences of pairs

$$(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j) \xrightarrow[\simeq]{pr_1} (\mathbb{R}^i, \mathbb{R}^i \setminus \{0\})$$
$$(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i) \xrightarrow[\simeq]{pr_2} (\mathbb{R}^j, \mathbb{R}^j \setminus \{0\}).$$

↗

Now consider

$$\begin{tikzcd}
H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j) \otimes H^j(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i) \arrow[rr, "\cup"] & & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\
 & \circlearrowleft & \\
H^i(\mathbb{R}^i, \mathbb{R}^i \setminus \{0\}) \otimes H^j(\mathbb{R}^j, \mathbb{R}^j \setminus \{0\}) \arrow[uu, "\cong"', "H^i(pr_1) \otimes H^j(pr_2)"] \arrow[uurr, bend right, "\times"', "\text{cohomological cross product}"] & &
\end{tikzcd}$$

last remark in 7.4.

$\Longrightarrow$ 1)

Ad 2) For ⓐ and ⓑ consider the diagram:

$$\begin{tikzcd}
H^i(P^n) \arrow[d, "\cong"', "\left(\substack{\text{from}\\\text{cellular}\\\text{homology}}\right)"] & H^i(P^n, P^{i-1}) \arrow[l, "\cong (LES)"'] \arrow[d, "\cong"', "\left(\substack{\text{from}\\\text{cellular}\\\text{homology}}\right)"] & H^i(P^n, P^n \setminus P^j) \arrow[l, "(*)"'] \arrow[r, "\text{chart}", "\cong"'] \arrow[d] & H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j) \arrow[d, "\cong"', "\substack{\text{induced by}\\\text{homotopy equivale}}"] \\
H^i(P^i) & H^i(P^i, P^{i-1}) \arrow[l, "\cong (LES)"'] & H^i(P^i, P^i \setminus \{p\}) \arrow[l, "(**)"'] \arrow[r, "\cong", "\substack{\text{restriction of}\\\text{chart homomorphism}}"'] & H^i(\mathbb{R}^i, \mathbb{R}^i \setminus \{0\})
\end{tikzcd}$$

by excision

The maps are induced by the inclusion unless otherwise stated.

⌈Remark to the lower excision-isomorphism:

Apply excision to $B = P^i \setminus \{p\}$, A = complement of $U \cap P^i$,

$$H^i(P, B) \cong H^i(P \setminus A, B \setminus A) .$$

Remark to the upper excision-isomorphism:

Apply excision to $B = P^n \setminus P^j$, $A = P^n \setminus U$. ⌋

(**) $P^{i-1} \hookrightarrow P^i \setminus \{p\}$ is a deformation retract

via $h_t([x_0, \dots, x_i]) = [x_0, \dots, x_{i-1}, t \cdot x_i]$.

LES + 5-lemma $\Rightarrow$ (**) is an isomorphism

(*) Similarly, $P^{i-1} \hookrightarrow P^n \setminus P^j$ is a deformation retract

via $h_t([x_0, \dots, x_n]) = [x_0, \dots, x_{i-1}, t x_i, \dots, t x_n]$. ↺

Finally, the middle vertical map is an isomorphism.

If we interchange i and j, then the analogous statement holds.

This implies that ⓐ and ⓑ are isomorphisms.

On ⓓ: follows from excision (cf. previous diagram)

On ⓒ:

$$\begin{tikzcd} H^n(P^n, P^n \setminus \{p\}) \arrow[rr, "ⓒ"] \arrow[dr, "\cong"] & & H^n(P^n) \\ & H^n(P^n, P^{n-1}) \arrow[ur, "\cong"'] & \end{tikzcd}$$

See ⊛⊛ with $i \leftrightarrow n$

see last diagram (left top for $i=n$)

▨

Complex projective spaces are handled in a similar way, replacing

$$\begin{cases} \mathbb{Z} \hookrightarrow \mathbb{F}_2 \\ H^i \hookrightarrow H^{2i} \\ \mathbb{R} \hookrightarrow \mathbb{C} \end{cases}.$$

The cases $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$ follow from the finite-dimensional statements since $\mathbb{R}P^i \hookrightarrow \mathbb{R}P^\infty$ induce H^*-isos up to degree i and $2i$,

$\mathbb{C}P^i \hookrightarrow \mathbb{C}P^\infty$

respectively. To sum up, we obtain:

Theorem:

$$H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad |\alpha| = 2,$$

$$H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha], \quad |\alpha| = 1,$$

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha], \quad |\alpha| = 2.$$

What about $H^*(\mathbb{R}P^n;\mathbb{Z})$?

A ring homomorphism $R \to S$ induces a homomorphism $H^*(X;R) \to H^*(X;S)$ of rings (even R-algebras).

Let's consider $H^*(\mathbb{R}P^\infty;\mathbb{Z}) \to H^*(\mathbb{R}P^\infty;\mathbb{F}_2)$ induced by the projection map $\mathbb{Z} \to \mathbb{F}_2$ on the level of cellular cochain complexes:

$$\begin{tikzcd}
\cdots & \mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l, "0"'] \arrow[d] & \mathbb{Z} \arrow[l, "2"'] \arrow[d] & \mathbb{Z} \arrow[l, "0"'] \arrow[d] & 0 \arrow[l] \\
\cdots & \mathbb{F}_2 \arrow[l] & \mathbb{F}_2 \arrow[l, "0"'] & \mathbb{F}_2 \arrow[l, "0"'] & \mathbb{F}_2 \arrow[l, "0"'] & 0 \arrow[l]
\end{tikzcd}$$

We see that $H^*(\mathbb{R}P^\infty;\mathbb{Z}) \to H^*(\mathbb{R}P^\infty;\mathbb{F}_2)$ is injective in positive degrees.

Moreover, $\operatorname{im}\left(H^*(\mathbb{R}P^\infty;\mathbb{Z}) \to H^*(\mathbb{R}P^\infty;\mathbb{F}_2)\right) = \bigoplus_{k\geq 0} H^{2k}(\mathbb{R}P^\infty;\mathbb{F}_2)$.

$\Rightarrow \quad H^*(\mathbb{R}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha)\ , \quad |\alpha| = 2$.

(As a module, $H^*(\mathbb{R}P^\infty;\mathbb{Z}) \cong \underset{0}{\mathbb{Z}} \oplus \underset{2}{\mathbb{F}_2\cdot\alpha} \oplus \underset{4}{\mathbb{F}_2\cdot\alpha^2} \oplus \underset{6}{\mathbb{F}_2\cdot\alpha^3} \oplus \underset{\cdots}{\cdots}$.)
deg:

Similarly, $H^*(\mathbb{R}P^{2k};\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha,\alpha^{k+1})\ , \quad |\alpha| = 2$,

since $H^*(\mathbb{R}P^\infty;\mathbb{Z}) \to H^*(\mathbb{R}P^{2k};\mathbb{Z})$ is an isomorphism in degrees $\leq 2k$.

For $\mathbb{R}P^{2k+1}$ the corresponding diagram of cellular cochain complexes is

$$\begin{tikzcd}
H^*(\mathbb{R}P^{2k+1};\mathbb{Z}) & 0 & \mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l, "0"'] \arrow[d] & \cdots \arrow[l, "2"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d] & 0 \arrow[l] \\
H^*(\mathbb{R}P^{2k+1};\mathbb{F}_2) & 0 & \underset{(2k+1)}{\mathbb{F}_2} \arrow[l] & \mathbb{F}_2 \arrow[l, "0"'] & \cdots \arrow[l, "0"'] & \mathbb{F}_2 \arrow[l, "0"'] & 0 \arrow[l]
\end{tikzcd}$$
.

↺

So
$$\begin{array}{ccl}
\mathbb{Z}[\alpha,\beta] & \longrightarrow & H^*(\mathbb{RP}^{2k+1};\mathbb{Z}) \\
\alpha & \longmapsto & \text{(unique) generator of } H^2(\mathbb{RP}^{2k+1};\mathbb{Z}) \cong \mathbb{F}_2 \\
\beta & \longmapsto & \text{a generator of } H^{2k+1}(\mathbb{RP}^{2k+1};\mathbb{Z}) \cong \mathbb{Z}
\end{array}$$

with $|\alpha| = 2$ and $|\beta| = 2k+1$ is surjective.

As a graded module, $H^*(\mathbb{RP}^{2k+1};\mathbb{Z})$ looks like

$$\begin{array}{lccccccccccccccc}
 & \langle\beta\rangle & & \langle\alpha^k\rangle & & & & \langle\alpha^{k-1}\rangle & & & & \langle\alpha\rangle & & & & \\
 & \shortparallel & & \shortparallel & & & & \shortparallel & & & & \shortparallel & & & & \\
 & \mathbb{Z} & \oplus & \mathbb{F}_2 & \oplus & 0 & \oplus & \mathbb{F}_2 & \oplus & \cdots & \oplus & \mathbb{F}_2 & \oplus & 0 & \oplus & \mathbb{Z} \\
\text{deg}: & 2k+1 & & 2k & & 2k-1 & & 2k-2 & & & & 2 & & 1 & & 0
\end{array}.$$

Hence, we get an isomorphism

$$H^*(\mathbb{RP}^{2k+1};\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}[\alpha,\beta] \big/ (2\alpha, \alpha^{k+1}, \alpha\beta, \beta^2), \quad \begin{array}{l} |\alpha| = 2 \\ |\beta| = 2k+1 \end{array}.$$

Remark:

$$H^*(\mathbb{RP}^{2k+1};\mathbb{Z}) \cong H^*(\mathbb{RP}^{2k} \vee S^{2k+1};\mathbb{Z}) \quad \text{as algebras},$$

but $\mathbb{RP}^{2k+1}$ and $\mathbb{RP}^{2k} \vee S^{2k+1}$ are not homotopy-equivalent.

Further, we have

$$H^*(\mathbb{RP}^{2k+1};\mathbb{F}_2) \cong H^*(\mathbb{RP}^{2k} \vee S^{2k+1};\mathbb{F}_2) \quad \text{as (graded) } \mathbb{F}_2\text{-modules.}$$

But $H^*(\mathbb{RP}^{2k+1};\mathbb{F}_2) \not\cong H^*(\mathbb{RP}^{2k} \vee S^{2k+1};\mathbb{F}_2)$ as $\mathbb{F}_2$-algebras, since the $(2k+1)$-th power of the generator of $H^1(\mathbb{RP}^{2k+1};\mathbb{F}_2)$ is non-zero while the $(2k+1)$-th power of the generator of $H^1(\mathbb{RP}^{2k} \vee S^{2k+1};\mathbb{F}_2)$ is zero (since it is zero in $H^{2k+1}(\mathbb{RP}^{2k};\mathbb{F}_2)$).

↗

7.6 Applications of the cup_product

The following map played an important role in the development of algebraic topology:

Def.: The map
$$\begin{array}{ccccc} \eta : & S^3 & \longrightarrow & \mathbb{C}\cup\{\infty\} & \overset{\text{stereographic projection}}{\cong} S^2 \\ & \cap\!\!\shortparallel & & & \\ & \mathbb{C}\times\mathbb{C} & & & \\ & (z_0,z_1) & \longmapsto & \frac{z_0}{z_1} & \end{array}$$

is called the Hopf_map_.

Let's try to understand better the geometry of η:

In polar coordinates, η looks like

$$\eta\left(r_0 e^{i\vartheta_0}, r_1 e^{i\vartheta_1}\right) = \frac{r_0}{r_1}\cdot e^{i(\vartheta_0-\vartheta_1)} \in \mathbb{C}\cup\{\infty\}.$$

Consider $C_r = \{z\in\mathbb{C} \mid |z|=r\} \subset \mathbb{C} \subset \mathbb{C}\cup\{\infty\}$.

Let
$$T_r := \eta^{-1}(C_r) = \left\{(r_0e^{i\vartheta_0}, r_1e^{i\vartheta_1}) \mid r_0^2+r_1^2=1,\ \frac{r_0}{r_1}=r\right\}$$
$$= \left\{(r_0e^{i\vartheta_0}, r_1e^{i\vartheta_1}) \mid r_0, r_1 \text{ fixed}\right\}.$$

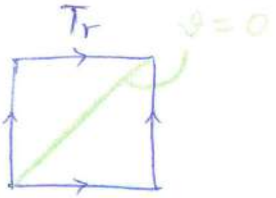
For $r>0$, T_r is a 2-torus.

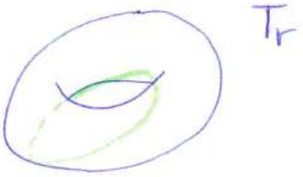
For $r=0$, T_0 is a circle, namely $T_0 = S^1\times\{0\}\subset S^3\subset\mathbb{C}\times\mathbb{C}$.

Set $T_\infty = \eta^{-1}(\{\infty\}) = \{0\}\times S^1 \subset S^3 \subset \mathbb{C}\times\mathbb{C}$.

Let $p\in C_r\subset\mathbb{C}\cup\{\infty\}$.

Then $\eta^{-1}(\{p\}) = \left\{(r_0e^{i\vartheta_0}, r_1e^{i\vartheta_1}) \;\middle|\; \begin{array}{l} r_0, r_1 \text{ fixed} \\ \vartheta_0-\vartheta_1=\vartheta \text{ fixed}\end{array}\right\} \cong S^1$.





↻

Think of $\underset{\mathbb{R}^4}{\overset{\cap\!\!|}{S^3}}$ as $\mathbb{R}^3 \cup \{\infty\}$ via stereographic projection

$$s: \; S^3 \xrightarrow{\cong} \mathbb{R}^3 \cup \{\infty\}$$

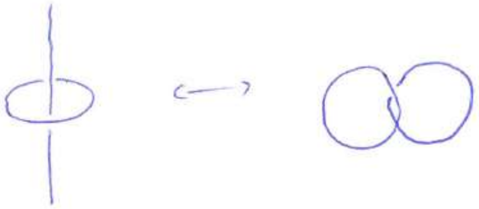
$$(x_1, \dots, x_4) \longmapsto \frac{1}{1-x_4}(x_1, x_2, x_3)$$

$$(0,0,0,1) \longmapsto \infty .$$

$s(T_0) = s(\{(x,y,0,0)\}) = \{(x,y,0)\}$ is the unit circle in the xy-plane of $\mathbb{R}^3 \subset \underset{S^3}{\underset{\shortparallel}{\mathbb{R}^3 \cup \{\infty\}}}$.

$s(T_\infty) = s(\{(0,0,x,y)\}) = \left\{ \frac{1}{1-y}(0,0,x) \right\}$ is the z-axis in $\mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\}$.

So the picture is :



7.6 Applications of the cup product

The following map played an important role in the development of algebraic topology:

Def.: The map

$$\begin{tikzcd} \eta : S^3 \arrow[r] & \mathbb{C} \cup \{\infty\} \overset{\text{stereographic projection}}{\cong} S^2 \\ \mathbb{C} \times \mathbb{C} \arrow[u, phantom, "\cap" rotate] & \\ (z_0, z_1) \arrow[r, mapsto] & \frac{z_0}{z_1} \end{tikzcd}$$

is called the Hopf map.

Let's try to understand better the geometry of η:

In polar coordinates, η looks like

$$\eta\left(r_0 e^{i\vartheta_0}, r_1 e^{i\vartheta_1}\right) = \frac{r_0}{r_1} \cdot e^{i(\vartheta_0 - \vartheta_1)} \in \mathbb{C} \cup \{\infty\} .$$

Consider $C_r = \left\{ z \in \mathbb{C} \mid |z| = r \right\} \subset \mathbb{C} \subset \mathbb{C} \cup \{\infty\}$.

Let $T_r := \eta^{-1}(C_r) = \left\{ (r_0 e^{i\vartheta_0}, r_1 e^{i\vartheta_1}) \mid r_0^2 + r_1^2 = 1 , \frac{r_0}{r_1} = r \right\}$

$$= \left\{ (r_0 e^{i\vartheta_0}, r_1 e^{i\vartheta_1}) \mid r_0, r_1 \text{ fixed} \right\} .$$

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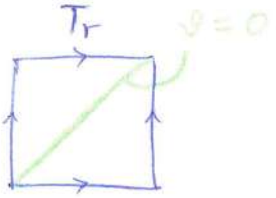
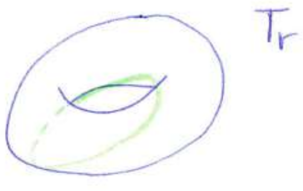
For $r = 0$, T_0 is a circle, namely $T_0 = S^1 \times \{0\} \subset S^3 \subset \mathbb{C} \times \mathbb{C}$.

Set $T_\infty = \eta^{-1}(\{\infty\}) = \{0\} \times S^1 \subset S^3 \subset \mathbb{C} \times \mathbb{C}$.

Let $p \in C_r \subset \mathbb{C} \cup \{\infty\}$.

Then $\eta^{-1}(\{p\}) = \left\{ (r_0 e^{i\vartheta_0}, r_1 e^{i\vartheta_1}) \;\middle|\; \begin{matrix} r_0, r_1 \text{ fixed} \\ \vartheta_0 - \vartheta_1 = \vartheta \text{ fixed} \end{matrix} \right\} \cong S^1$.





↺

Think of $\underset{\underset{\mathbb{R}^4}{\cap\!\shortmid}}{S^3}$ as $\mathbb{R}^3 \cup \{\infty\}$ via stereographic projection

$$s: S^3 \xrightarrow{\cong} \mathbb{R}^3 \cup \{\infty\}$$

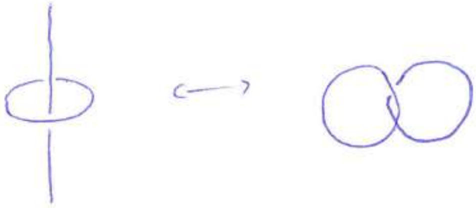
$$(x_1, \dots, x_4) \longmapsto \frac{1}{1-x_4}(x_1, x_2, x_3)$$

$$(0,0,0,1) \longmapsto \infty .$$

$s(T_0) = s(\{(x,y,0,0)\}) = \{(x,y,0)\}$ is the unit circle in the xy-plane of $\mathbb{R}^3 \subset \underset{\underset{S^3}{\shortparallel}}{\mathbb{R}^3 \cup \{\infty\}}$.

$s(T_\infty) = s(\{(0,0,x,y)\}) = \left\{\frac{1}{1-y}(0,0,x)\right\}$ is the z-axis in $\mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\}$.

So the picture is:



7.12.

Recall the Hopf map $\eta: \underset{\underset{\mathbb{C}\times\mathbb{C}}{\cap\!\shortmid}}{S^3} \longrightarrow \underset{\underset{\mathbb{C}\cup\{\infty\}}{\shortparallel}}{S^2}$.

$$(z_0, z_1) \longmapsto \frac{z_0}{z_1}$$

<u>Theorem</u>: The Hopf map is <u>not</u> null-homotopic.

First we describe a general construction to see that a map $f: X \longrightarrow Y$ is not null-homotopic.

Recall : $\quad \mathrm{Cone}(X) = {X \times [0,1]}/{X \times \{1\}}$

The <u>mapping cone</u> of f is defined by

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "j"] \\ \mathrm{cone}(X) \arrow[r, "q"'] & \mathrm{cone}(f) \end{tikzcd} .$$

Say f was null-homotopic, then there is $y_0 \in Y$ and a homotopy $h : [0,1] \times X \longrightarrow Y$ with $h_0 = f$ and $h_1 \equiv y_0$.

It follows that $j : Y \hookrightarrow \operatorname{cone}(f)$ is a retract.

The map $r : \operatorname{cone}(f) \longrightarrow Y$,

$$r(z) = \begin{cases} h_t(x) & , \text{ if } \quad z = q([x,t]) \in q(\operatorname{cone}(f)), \\ z & , \quad \text{otherwise}, \end{cases}$$

satisfies $r \circ j = id_Y$.

Contrapositive : If j is not a retract, then f is not null-homotopic.

Now let $f = \eta$ be the Hopf map.

Recall the standard CW structure on $\mathbb{C}P^n$. The 4-skeleton is obtained by

$$\begin{tikzcd} S^3 \arrow[r, "p"] \arrow[d, hook] & \mathbb{C}P^1 \arrow[d] \\ D^4 \arrow[r] & \mathbb{C}P^2 \end{tikzcd} \qquad \begin{tikzcd} S^3 \arrow[r, "p"] \arrow[d, hook] & \mathbb{C}P^1 \arrow[d, equal] \\ \mathbb{C}\times\mathbb{C} \arrow[r] & \mathbb{C}P^1 \\ (z_0, z_1) \arrow[r, mapsto] & {[z_0, z_1]} \end{tikzcd} .$$

Note that $\mathbb{C}P^1 \cong \underset{(\mathbb{C}\cup\{\infty\})}{\underset{\cong}{S^2}}$ via $[z_0 : z_1] \longmapsto \frac{z_0}{z_1} \longmapsto s^{-1}\left(\frac{z_0}{z_1}\right)$,

where $s : S^2 \to \mathbb{C} \cup \{\infty\}$ is the stereographic projection.

We denote this composition by g and get $g \circ p = \eta$. Consider

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d] \arrow[rr, bend left, "\eta"] & \mathbb{C}P^1 \arrow[r, "\cong", "g"'] \arrow[d, "\text{incl}"] & S^2 \arrow[d, "\text{incl} \circ g^{-1}"] \\ D^4 \arrow[r] & \mathbb{C}P^2 \arrow[r, "id"] & \mathbb{C}P^2 \end{tikzcd}$$

↺

The outer square is a pushout. Since $D^4 = \text{cone}(S^3)$, we have $\mathbb{C}P^2 = \text{cone}(\eta)$.

Thus it remains to show that $\mathbb{C}P^1 \xhookrightarrow{\text{incl}} \mathbb{C}P^2$ is not a retract.

Therefore, we will use the cup product.

Let $z \in H^2(\mathbb{C}P^2)$ and $y \in H^2(\mathbb{C}P^1)$ be generators.

Then $H^2(\text{incl})(z) = \pm y$. $\leftarrow$ H^2(incl) surjective

If there was a retract $r: \mathbb{C}P^2 \to \mathbb{C}P^1$, then we would get a homomorphism $H^*(r): H^*(\mathbb{C}P^1) \longrightarrow H^*(\mathbb{C}P^2)$ of $\mathbb{Z}$-graded algebras,

$$\begin{tikzcd} H^*(\mathbb{C}P^1) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^*(\mathbb{C}P^2) \arrow[d, phantom, "\cong" rotate=90] \\ \varphi: \mathbb{Z}[y]/(y^2) \arrow[r] & \mathbb{Z}[z]/(z^3) \end{tikzcd}$$

sending y to $\pm z$. This leads to

$$0 = \varphi(0) = \varphi(y \cup y) = \varphi(y) \cup \varphi(y) = z \cup z \neq 0 ,$$

which is a contradiction. ☒

There are analogous constructions as the Hopf map for quaternions and octonions.

The quaternions $\mathbb{H}$ are a 4-dim. non-commutative $\mathbb{R}$-algebra; as a vector space $\mathbb{H}$ is generated by e, I, j, k.

Multiplication is defined by

$$\begin{tikzcd} & I \arrow[dr] & \\ k \arrow[ur] & & j \arrow[ll] \end{tikzcd}, \quad I \cdot j = k,\ j \cdot k = I,\ k \cdot I = j, \quad I^2 = j^2 = k^2 = -1 \cdot e.$$

$$\Rightarrow j \cdot I = -j(-1)I = -(j \cdot k)(k \cdot I) = -I \cdot j$$

We have a quaternionic conjugation sending

$$e \mapsto e, \quad I \mapsto -I, \quad j \mapsto -j, \quad k \mapsto -k.$$

↗

The octonions (aka Cayley numbers) $\mathbb{O} = \mathbb{H}\times\mathbb{H}$ form an 8-dimensional $\mathbb{R}$-vector space with multiplication

$$\mathbb{O}\times\mathbb{O}\longrightarrow\mathbb{O}$$

$$\big((a_1,a_2),(b_1,b_2)\big)\longmapsto\left(a_1b_1-\bar{b}_2a_2\,,\ a_2\bar{b}_1+b_2a_1\right)$$

This multiplication is not associative.

But $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are division algebras.

Def.: A real division algebra structure on $\mathbb{R}^n$ is a bilinear map ("product") $\mu: \mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that:

1) μ is distributive,

$$a(b+c) = ab+ac \quad \forall a,b,c\in\mathbb{R}^n$$
$$(b+c)a = ba+ca \quad \forall a,b,c\in\mathbb{R}^n$$

2) μ is scalar associative:

$$r(a\cdot b) = (ra)b = a(rb) \qquad \forall r\in\mathbb{R}\quad \forall a,b\in\mathbb{R}^n$$

3) There is $e\in\mathbb{R}^n$ with $a\cdot e = a = e\cdot a$.

4) The maps $\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $x\longmapsto a\cdot x$
and $\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $x\longmapsto x\cdot a$
are bijective $\forall a\in\mathbb{R}^n\setminus\{0\}$.

Def.: The quaternionic Hopf map is $\nu:\ \underset{\mathbb{H}\times\mathbb{H}}{\underset{\cap}{S^7}}\longrightarrow\underset{\mathbb{H}\cup\{\infty\}}{\underset{\|}{S^4}}$,

$$(z_0,z_1)\longmapsto\frac{z_0}{z_1}.$$

Def.: The octonionic Hopf map is $\sigma:\ \underset{\mathbb{O}\times\mathbb{O}}{\underset{\cap}{S^{15}}}\longrightarrow\underset{\mathbb{O}\cup\{\infty\}}{\underset{\|}{S^8}}$,

$$(z_0,z_1)\longmapsto\frac{z_0}{z_1}.$$

Similarly as for η, one sees that ν and σ are not null-homotopic. One uses

$\mathbb{H}P^2$ with $H^*(\mathbb{H}P^2) \cong \mathbb{Z}[w]/(w^3)$, $|w| = 4$, and

$\mathbb{O}P^2$, defined by

$$\begin{tikzcd} S^{15} \arrow[r, "\sigma"] \arrow[d] & S^8 \arrow[d] \\ D^{16} \arrow[r] & \mathbb{O}P^8 \end{tikzcd}.$$

Caution: There are no $\mathbb{O}P^n$ for $n > 2$.

One needs associativity to show that $(z_0, \dots, z_n) \sim \underset{\underset{\mathbb{O}\setminus\{0\}}{\uparrow}}{\lambda} (z_0, \dots, z_n)$ defines an equivalence relation.

Question: Do division algebra structures exists on $\mathbb{R}^n$ for $n \notin \{1, 2, 4, 8\}$?

Theorem (Hopf): If there exists a division algebra structure on $\mathbb{R}^n$, then n is a power of 2.

Proof: Assume $n > 2$ and let $\mu: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(a,b) \mapsto a \cdot b$ be a division algebra structure.

Define $g: S^{n-1} \times S^{n-1} \to S^{n-1}$, $(x,y) \mapsto \frac{x \cdot y}{\|x \cdot y\|}$.

We have $g(-x,y) = g(x,-y) = -g(x,y)$ for all $x,y \in S^{n-1}$ and thus g induces a map $h: \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$.

Let $\gamma \in H^1(\mathbb{R}P^{n-1}, \mathbb{F}_2)$ be a generator of $H^*(\mathbb{R}P^{n-1}; \mathbb{F}_2) \cong \mathbb{F}_2[\gamma]/(\gamma^n)$.

Let $pr_i: \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$ for $i = 1,2$ be the projections.

Set $\alpha = pr_1^*(\gamma) \in H^1(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{F}_2)$

and $\beta = pr_2^*(\gamma) \in$ — " — .

Claim: $h^*(\gamma) = \alpha + \beta$.

If we take the claim for granted, then

$$0 = h^*(0) = h^*(\gamma^n) = (h^*\gamma)^n = (\alpha+\beta)^n \in H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{F}_2)$$

$$\text{Künneth} \rightarrow \cong H^*(\mathbb{R}P^{n-1}; \mathbb{F}_2) \underset{\mathbb{F}_2}{\otimes} H^*(\mathbb{R}P^{n-1}; \mathbb{F}_2) .$$

Moreover,

$$H^*(\mathbb{R}P^{n-1}; \mathbb{F}_2) \underset{\mathbb{F}_2}{\otimes} H^*(\mathbb{R}P^{n-1}; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha]/(\alpha^n) \underset{\mathbb{F}_2}{\otimes} \mathbb{F}_2[\beta]/(\beta^n) \cong \mathbb{F}_2[\alpha,\beta]/(\alpha^n, \beta^n) .$$

$$\Longrightarrow \quad \binom{n}{k} \equiv 0 \pmod 2 \quad \text{for } 0 < k < n .$$

number theory $\rightsquigarrow$ n is a power of two.

It remains to prove: 8.12.

a) $H^1(h)(\gamma) = \alpha + \beta \in H^1(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{F}_2)$

b) $\binom{n}{k} = 0 \pmod 2$ for all $k \in \{1, \dots, n-1\}$.

$\iff$ n is a power of two

Ad a): Consider

$$\begin{tikzcd}
S^{n-1} \arrow[r, "j_1"] \arrow[d, "p"] \arrow[dr, phantom, "\circlearrowleft"] & S^{n-1} \times S^{n-1} \arrow[r, "g"] \arrow[d, "p \times p"] \arrow[dr, phantom, "\circlearrowleft"] & S^{n-1} \arrow[d, "\text{2-fold covering}"] \\
\mathbb{R}P^{n-1} \arrow[r, "\overline{j_1}"] & \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \arrow[r, "h"] & \mathbb{R}P^{n-1}
\end{tikzcd}$$

,

where $j_1(x) = (x, x_0)$ for a basepoint $x_0 \in S^{n-1}$.

Assume from now on that $n > 2$.

Let $\alpha : [0,1] \longrightarrow S^{n-1}$ be a path joining x_0 and $-x_0$.
Then
$$[p\circ\alpha] \in \pi_1(\mathbb{R}P^{n-1}) \cong \mathbb{Z}/2 \quad \text{is the generator.}$$

Since $g\circ j_1\circ\alpha$ joins $g(x_0,x_0)$ and $g(-x_0,x_0) = -g(x_0,x_0)$,
the element $[p\circ g\circ j_1\circ\alpha] = [h\circ\bar{j}_1\circ(p\circ\alpha)] = \pi_1(h\circ\bar{j}_1)([p\circ\alpha]) \in \pi_1(\mathbb{R}P^{n-1})$
is the generator. Hence

$$\pi_1(h\circ j_1) : \pi_1(\mathbb{R}P^{n-1}) \longrightarrow \pi_1(\mathbb{R}P^{n-1})$$

is the identity. By Hurewicz, $H_1(h\circ\bar{j}_1)$ is the identity.

By UCT, $H^1(h\circ\bar{j}_1)$ is the identity, too.

The same argument holds for the embedding j_2 into the 2nd factor.

By the Künneth theorem,

$$H^1(\mathbb{R}P^{n-1};\mathbb{F}_2) \oplus H^1(\mathbb{R}P^{n-1};\mathbb{F}_2) \xrightarrow[H^1(pr_1)+H^1(pr_2)]{\cong} H^1(\mathbb{R}P^{n-1}\times\mathbb{R}P^{n-1};\mathbb{F}_2) \qquad (*)$$

is an isomorphism:

$$\begin{tikzcd}
 & H^1(\mathbb{R}P^{n-1}\times\mathbb{R}P^{n-1};\mathbb{F}_2) \\
x\otimes 1 + 1\otimes y & H^1(\mathbb{R}P^{n-1};\mathbb{F}_2)\otimes_{\mathbb{F}_2} H^0(\mathbb{R}P^{n-1};\mathbb{F}_2) \oplus H^0(\mathbb{R}P^{n-1};\mathbb{F}_2)\otimes_{\mathbb{F}_2} H^1(\mathbb{R}P^{n-1};\mathbb{F}_2) \arrow[u, "\times"', "\cong\ (\text{Künneth})"] \\
(x,y) \arrow[u, mapsto] & H^1(\mathbb{R}P^{n-1};\mathbb{F}_2)\oplus H^1(\mathbb{R}P^{n-1};\mathbb{F}_2) \arrow[u, "\cong"']
\end{tikzcd}$$

The composition yields
$$(x,0) \longmapsto (x\otimes 1, 0) \longmapsto x\times 1 = H^1(pr_1)(x) \cup \overbrace{H^1(pr_2)(1)}^{=1} = H^1(pr_1)(x)$$
and similarly
$$(0,y) \longmapsto H^1(pr_2)(y).$$

↗

The inverse of $(*)$ is $w \longmapsto \left(H^1(\bar{j}_1)(w),\ H^1(\bar{j}_2)(w) \right)$.

For example,

$$\left(H^1(pr_1) + H^1(pr_2) \right)\left(H^1(\bar{j}_1)(w),\ H^1(\bar{j}_2)(w) \right) = H^1(\underbrace{\bar{j}_1 \circ pr_1}_{(id,\, const.)})(w) + H^1(\underset{(const,\, id)}{\bar{j}_2 \circ pr_2})(w)$$

$$= w \quad \text{for } w \in H^1(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1};\ \mathbb{F}_2).$$

It is enough to verify that

$$H^1(j_k)(\alpha+\beta) \overset{!}{=} H^1(\bar{j}_k)\left(H^1(h)(\gamma) \right) \quad \text{for } k \in \{1,2\}.$$

LHS:
$$H^1(\bar{j}_1)(\alpha+\beta) = H^1(\bar{j}_1)\left(H^1(pr_1)(\gamma) \right) + H^1(\bar{j}_1)\left(H^1(pr_2)(\gamma) \right)$$

$$= H^1(\underbrace{pr_1 \circ \bar{j}_1}_{= id})(\gamma) + H^1(\underbrace{pr_2 \circ \bar{j}_1}_{\text{constant map}})(\gamma)$$

$$= \gamma + 0 = \gamma.$$

Similarly, $H^1(\bar{j}_2)(\alpha+\beta) = 0 + \gamma = \gamma$.

RHS:
$$H^1(\bar{j}_k)\left(H^1(h)(\gamma) \right) = \underbrace{H^1(h \circ \bar{j}_k)}_{\substack{= id \\ \text{(see above)}}}(\gamma) = \gamma.$$

$\Rightarrow$ a).

Ad b): First observe that

$$\binom{n}{k} \equiv 0 \pmod 2 \quad \text{for } k \in \{1, \dots, n-1\}$$

$$\Updownarrow$$

$$(1+X)^n = 1 + X^n \quad \text{in } \mathbb{F}_2[X].$$

Write $n = n_1 + \dots + n_k$ with $n_1 < n_2 < \dots$, n_i power of 2 (binary representation). Assume $(1+X)^n = 1 + X^n$ in $\mathbb{F}_2[X]$. ↺

$$1 + x^n = (1+x)^{n_1} \cdots (1+x)^{n_k}$$

$$= (1+x^{n_1}) \cdots (1+x^{n_k})$$

$\overset{\text{coefficients}}{\Rightarrow}$ $k=1$, $n=n_1$, so n is a power of 2. ☒

7.7 Cap - Product

Def.: Let X be a top. space.

Let R be a commutative ring.

The cap product (on cochains) is defined as

$$C^{\mathrm{sing}}_{k+\ell}(X;R) \underset{R}{\otimes} \underset{\underset{\hom_R(C^{\mathrm{sing}}_\ell(X;R);R)}{\shortparallel}}{C^\ell_{\mathrm{sing}}(X;R)} \xrightarrow{\cap} C^{\mathrm{sing}}_k(X;R)$$

$$\sigma \otimes \varphi \longmapsto \varphi\left(\sigma|_{[e_1,\dots,e_{\ell+1}]}\right) \cdot \sigma|_{[e_{\ell+1},\dots,e_{\ell+k+1}]} .$$

One easily verifies that

$$\partial\sigma \cap \varphi = \sigma \cap \delta\varphi + (-1)^k \partial(\sigma \cap \varphi)$$

for $\sigma \in C^{\mathrm{sing}}_{k+\ell}(X;R)$ and $\varphi \in C^\ell_{\mathrm{sing}}(X;R)$.

Hence $\cap$ on cochains induces the <u>cap product</u> on (Co-)homology:

$$H_{k+\ell}(X;R) \underset{R}{\otimes} H^\ell(X;R) \xrightarrow{\cap} H_k(X;R) .$$

For a pair (X,A) we obtain similarly:

$$H_{k+\ell}(X,A;R) \underset{R}{\otimes} H^\ell(X;R) \xrightarrow{\cap} H_k(X,A;R)$$

$$H_{k+\ell}(X,A;R) \underset{R}{\otimes} H^\ell(X,A;R) \xrightarrow{\cap} H_k(X;R)$$

↗

By linearity we obtain

$$\underbrace{H_*(X,A;R)}_{\substack{\| \\ \bigoplus_{p\geq 0} H_p(X,A;R)}} \underset{R}{\otimes} \underbrace{H^*(X;R)}_{\substack{\| \\ \bigoplus_{p\geq 0} H^p(X;R)}} \longrightarrow H_*(X,A;R) .$$

<u>Prop.:</u> $H_*(X,A;R)$ becomes a right module over the ring $H^*(X;R)$ via the cap product.

This module structure is natural in the following sense:

Let $f:(X,A) \longrightarrow (Y,B)$ be a map.

Then $H_*(f): H_*(X,A) \longrightarrow H_*(Y,B)$ is a homomorphism of $H_*(Y;R)$ - modules, where the $H^*(Y;R)$ - module structure on $H_*(X,A;R)$ is given by

$$x \cdot \varphi := x \cap \underbrace{H^*(f)(\varphi)}_{\in H^*_\bullet(X;R)} .$$

Explicitly, we have for $x \in H_{k+\ell}(X,A;R)$ and $\alpha \in H^\ell(Y;R)$:

$$H_k(f)\Big(x \cap H^\ell(f)(\alpha)\Big) = H_{k+\ell}(f)(x) \cap \alpha .$$

By linearity we obtain

$$\underbrace{H_*(X,A;R)}_{\overset{\shortparallel}{\bigoplus_{p\geq 0} H_p(X,A;R)}} \underset{R}{\otimes} \underbrace{H^*(X;R)}_{\overset{\shortparallel}{\bigoplus_{p\geq 0} H^p(X;R)}} \longrightarrow H_*(X,A;R) .$$

$H_* \otimes H^* \to H_*$

Prop.: $H_*(X,A;R)$ becomes a right module over the ring $H^*(X;R)$ via the cap product.

$H_*(X,A)$ ist $H^*(Y)$-Modul!

This module structure is natural in the following sense:

Let $f: (X,A) \longrightarrow (Y,B)$ be a map.

Then $H_*(f): H_*(X,A) \longrightarrow H_*(Y,B)$ is a homomorphism of $H_*(Y;R)$-modules, where the $H^*(Y;R)$-module structure on $H_*(X,A;R)$ is given by

$$x \cdot \varphi := x \cap \underbrace{H^*(f)(\varphi)}_{\in H^*_*(X;R)} .$$

Explicitly, we have for $x \in H_{k+\ell}(X,A;R)$ and $\alpha \in H^\ell(Y;R)$:

$$H_k(f)\Big(x \cap H^\ell(f)(\alpha)\Big) = H_{k+\ell}(f)(x) \cap \alpha .$$

14.12.

Proof: On being a right module over $H^*(X;R)$:

Let $\sigma \in C^{sing}_{k+\ell}(X,A;R)$, $\varphi \in C^p_{sing}(X;R)$ and $\psi \in C^q_{sing}(X;R)$ with $p+q=\ell$.

To show: $(\sigma \cap \varphi) \cap \psi = \sigma \cap (\varphi \cup \psi) \in C^{sing}_k(X,A;R)$

$$\sigma \cap \varphi = \varphi\left(\sigma_{|[e_1,\dots,e_{p+1}]}\right) \cdot \sigma|_{[e_{p+1},\dots,e_{k+\ell+1}]}$$

$$(\sigma \cap \varphi) \cap \psi = \varphi\left(\sigma|_{[e_1,\dots,e_{p+1}]}\right) \cdot \left(\psi\left(\sigma|_{[e_{p+1},\dots,e_{\ell+1}]}\right) \cdot \sigma|_{[e_{\ell+1},\dots,e_{k+\ell+1}]}\right) \circlearrowleft$$

$$(\varphi \cup \psi)(\rho) = \varphi\left(\rho|_{[e_1,\dots,e_{p+1}]}\right) \cdot \psi\left(\rho|_{[e_{p+1},\dots,e_{\ell+1}]}\right)$$

$$\sigma \cap (\varphi \cup \psi) = (\varphi \cup \psi)\left(\sigma|_{[e_1,\dots,e_{\ell+1}]}\right) \cdot \sigma|_{[e_{\ell+1},\dots,e_{k+\ell+1}]}$$

Further, $\sigma \cap 1_X = \sigma$ is clear.

The homomorphism property also holds on the cochain level:

Consider singuler simplex $\sigma \in C^{sing}_{k+\ell}(X,A;R)$ and $\alpha \in C^{\ell}_{sing}(Y;R)$.

To show: $C^{sing}_k(f)\left(\sigma \cap C^{\ell}_{sing}(f)(\alpha)\right) = \alpha\left((f \circ \sigma)|_{[e_1,\dots,e_{\ell+1}]}\right)(f\circ\sigma)|_{[e_{\ell+1},\dots,e_{k+\ell+1}]}$

LHS: $$C^{sing}_k(f)\left(\sigma \cap C^{\ell}_{sing}(f)(\alpha)\right) = C^{sing}_k(f)\left(\alpha\left(f\circ\sigma|_{[e_1,\dots,e_{\ell+1}]}\right)\cdot \sigma|_{[e_{\ell+1},\dots,e_{k+\ell+1}]}\right)$$

$$= \alpha\left(f\circ\sigma|_{[e_1,\dots,e_{\ell+1}]}\right)\ f\circ\sigma|_{[e_{\ell+1},\dots,e_{k+\ell+1}]}$$

RHS: $$\alpha\left((f\circ\sigma)|_{[e_1,\dots,e_{\ell+1}]}\right)\cdot (f\circ\sigma)|_{[e_{\ell+1},\dots,e_{k+\ell+1}]}$$

$$= \alpha\left(f\circ\sigma|_{[e_1,\dots,e_{\ell+1}]}\right)\cdot f\circ\sigma|_{[e_{p+1},\dots,e_{k+\ell+1}]}$$

The natural homomorphism

$$H^i(X,A;R) \longrightarrow \hom_R(H_i(X,A;R), R)$$

yields a bilinear form, and therefore a homomorphism (evaluation)

$$\begin{aligned} H^i(X,A;R) \otimes_R H_i(X,A;R) &\longrightarrow R \\ x \otimes y &\longmapsto \langle x, y\rangle \end{aligned}$$

which is called the "<u>Kronecker product</u>".

$\nearrow$

Prop.: For $u \in H^p(X,A;R)$, $v \in H^q(X,A;R)$ and $w \in H_{p+q}(X,A;R)$ we have the relation $\langle u \cup v, w \rangle = \langle u, w \cap v \rangle$.

Proof: Straightforward verification on the (co-)chain level. $\boxtimes$

8. Vector bundles

8.1 Basic notions:

Def.: A real (or complex) n-dimensional vector bundle over a topological space B consists of a continuous map $p: E \to B$ and the structure of an n-dimensional real (or complex) vector space on each fiber $E_b := p^{-1}(b)$, $b \in B$, s.th. p is locally trivial in the following sense:

For every $b \in B$ there is an open neighborhood U of $b \in B$ and a commutative diagram

$$\begin{tikzcd} p^{-1}(U) =: E|_U \arrow[r, "\varphi", "\cong"'] \arrow[d, "p|_U"'] & U \times \mathbb{R}^n \arrow[d] \\ U \arrow[r, "="] & U \end{tikzcd}$$

($U \times \mathbb{C}^n$ for complex v.b.)

with a homeomorphism φ that maps fibers linear to fibers. A 1-dimensional vector bundle is called a line bundle.

Example: Trivial vector bundle

$$\begin{tikzcd} B \times \mathbb{R}^n \arrow[d, "pr_1"] \\ B \end{tikzcd}$$

↺

Example : Tautological line bundle over projective spaces $\mathbb{R}P^n$.

Set

$$\begin{tikzcd}
E := \left\{ \left([x_0, \dots, x_n], v \right) \,\middle|\, v \in \operatorname{span}\{(x_0, \dots, x_n)\} \right\} \subseteq \mathbb{R}P^n \times \mathbb{R}^{n+1} \arrow[d, "p" , "\text{projection to first coordinate}"'] \\
\mathbb{R}P^n
\end{tikzcd}$$

Consider $[x_0, \dots, x_n] \in \mathbb{R}P^n$. Say $x_i \neq 0$ and $U = \left\{ [y_0, \dots, y_n] \in \mathbb{R}P^n \mid y_i \neq 0 \right\}$.

Recall that U is the domain of the manifold chart

$$\begin{aligned} U &\xrightarrow{\cong} \mathbb{R}^n \\ [y_0, \dots, y_n] &\longmapsto \left(\frac{y_0}{y_i}, \dots, \hat{y_i}, \dots, \frac{y_n}{y_i} \right). \end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{R}^{n+1}$.

Consider
$$\begin{tikzcd}
E_{|U} \arrow[r, "\varphi", "\cong"'] \arrow[d, "p|_U"'] & U \times \mathbb{R} \arrow[d] \\
U \arrow[r, "="] & U
\end{tikzcd}$$

with $$\varphi\left(\left([y_0, \dots, y_n], v \right) \right) = \left([y_0, \dots, y_n], \left\langle v, \left(\frac{y_0}{y_i}, \dots, \overset{\substack{\text{i-th spot} \\ \downarrow}}{1}, \dots, \frac{y_n}{y_i} \right) \right\rangle \right).$$

φ is clearly fiberwise linear. The inverse is given by :

$$\left([y_0, \dots, y_n], s \right) \longmapsto \left([y_0, \dots, y_n], s \cdot \frac{1}{\left\| \left(\frac{y_0}{y_i}, \dots, \frac{y_n}{y_i} \right) \right\|} \left(\frac{y_0}{y_i}, \dots, \frac{y_n}{y_i} \right) \right)$$

↗

Def.: A map of vector bundles consists of a commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p_E"'] & F \arrow[d, "p_F"] \\ B \arrow[r] & C \end{tikzcd}$$

where p_E and p_F are vector bundles and the map on the top is fiberwise linear.

In the case where $B = C$ and the lower map is id_B we call this a bundle map over B.

The vector bundles over B with bundle maps over B as morphisms forms a category.

Prop.: A bundle map

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr] & & F \arrow[dl] \\ & B & \end{tikzcd}$$

such that g is a linear isomorphism on each fiber is an isomorphism of vector bundles, i.e. an isomorphism in the category above.

Proof: Since g is bijective, it suffices to show that g is locally a homeomorphism. Hence we may assume that E and F are trivial $E = F = U \times \mathbb{R}^n$.

$g : U \times \mathbb{R}^n \longrightarrow U \times \mathbb{R}^n$ is of the form $(x, v) \longmapsto (x, A(x)v)$

where $A : U \longrightarrow GL(n, \mathbb{R})$ is continuous.

The inverse is $(x, v) \longmapsto (x, A(x)^{-1} v)$.

It is continuous since $GL(n,\mathbb{R}) \xrightarrow{\text{inverse}} GL(n,\mathbb{R})$ is continuous.

Thus, $$\begin{tikzcd} U \arrow[r] & GL(n,\mathbb{R}) \arrow[r, "\text{inverse}"] & Gl(n,\mathbb{R}) \\ x \arrow[r, mapsto] & A(x) \arrow[r, mapsto] & A^{-1}(x) \end{tikzcd}$$ is continuous.

☒

Def.: A sequence of bundle maps is exact if it is fiberwise exact as a sequence of vector spaces.

Theorem: Let
$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p_E"'] & & F \arrow[dl, "p_F"] \\ & B & \end{tikzcd}$$
be a bundle map such that the rank of $g_x : E_x \to F_x$, $x \in B$, is constant.

Then the following hold:

1) $\ker(g) := \bigcup_{b \in B} \ker(g_b)$ is a vector bundle over B (with respect to p_E)

2) $\operatorname{Im}(g) := \bigcup_{b \in B} \operatorname{Im}(g_b)$ ———— " ———— (—— " —— p_F).

3) $\operatorname{Coker}(g) = \bigcup_{b \in B} \operatorname{coker}(g_b)$ ———— " ———— with respect to the map induced by p_F and the quotient topology.

Rem.: Consider
$$\begin{tikzcd} g: [0,1] \times \mathbb{R}^n \arrow[r] \arrow[d] & {[0,1] \times \mathbb{R}^n} \arrow[d] \\ {[0,1]} \arrow[r, "="] & {[0,1]} \end{tikzcd}$$
, $g(t,v) = (t, t \cdot v)$.

g does not satisfy the rank-condition.

$\ker(g)$ is not a vector bundle since the fiber over 0 is n-dim. and the fiber over 1 is 0-dim.

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15.

Proof of the theorem:

It suffices to prove local triviality. For that we may assume that $g: E \to F$ is a bundle map between trivial vector bundles $E = B \times \mathbb{R}^m$, $F = B \times \mathbb{R}^n$. Fix $b \in B$.

Let $K := \ker(g_b) \subseteq \mathbb{R}^m$, $L := \operatorname{im}(g_b) \subseteq \mathbb{R}^n$.

Choose linear projections $p: \mathbb{R}^m \to K$, $q: \mathbb{R}^n \to L$.

The map $f_x: \mathbb{R}^m \longrightarrow K \oplus L$ is a linear isomorphism at $x = b$.

By a choice of basis for $K \oplus L$ we can consider the continuous function $x \longmapsto \det(f_x)$. This function is non-zero at $x = b$, hence it is non-zero in some neighborhood V of b. In other words, f_x is a linear isomorphism for $x \in V$.

For all x, $f_x(\ker g_x) \subseteq K = K \oplus \{0\}$.

By assumption, $\dim(\ker g_x) = \dim(\ker g_b) = \dim K$.

Hence for $x \in V$, $f_x: \ker(g_x) \to K$ is a linear isomorphism.

Thus
$$\begin{tikzcd}
\ker g|_V \arrow[r] \arrow[d] & V \times K \arrow[d] \\
V \arrow[r, "="] & V
\end{tikzcd}$$
with $\underset{\ker(g_x)}{\overset{\Uparrow}{\varphi(w)}} = (x, f_x(w))$

shows triviality of $\ker(g)|_V$.

The other cases are similar. ▨

↺

Prop. and Def.:

Let $E \xrightarrow{p} B$ be a vector bundle and $f: C \to B$.

The pullback f^*E

$$\begin{tikzcd} f^*E \arrow[r, dashed] \arrow[d, dashed] & E \arrow[d, "p"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

has the structure of a vector bundle over C such that the structure map $f^*E \to E$ is a map of vector bundles (in fact, consisting of isos in each fiber).

Such a structure is unique up to canonical isomorphism.

Proof:

$$\begin{tikzcd} f^*E = \{(c,v) \mid f(c) = p(v)\} \subset C \times E \arrow[d, "\text{projection onto the first coordinate}"] \\ C \end{tikzcd}$$

Canonically $(f^*E)_c = E_{f(c)}$ which declares the structure of a vector space on each fiber of f^*E.

The structure map $f^*E \to E$ is the identity $(f^*E)_c = E_{f(c)} \to E_{f(c)}$ in each fiber.

f^*E satisfies local triviality: Let $c \in C$. Let U be a neighborhood of $f(c)$ s.t. $E|_U$ is trivial. Now consider

$$\begin{tikzcd} f^*E|_{f^{-1}(U)} \arrow[r, "\varphi"] \arrow[d] & f^{-1}(U) \times \mathbb{R}^n \arrow[d] \\ f^{-1}(U) \arrow[r, "="] & f^{-1}(U) \end{tikzcd} \qquad \left(\begin{tikzcd} E_U \arrow[r, "\psi", "\cong"'] \arrow[d] & U \times \mathbb{R}^n \arrow[d] \\ U \arrow[r, "="] & U \end{tikzcd} \right)$$

where $\varphi((c,v)) = (c, (pr_2 \circ \psi)(v))$. ↗

The pullback yields a functor from the category of vector bundles over B to the category of vb over C.

8.2 Further constructions of vector bundles

Every standard construction in linear algebra can be transferred to vector bundles by applying it fiber wise.

Examples:

V^* dual space		E^* dual bundle
$V \oplus W$		$E \oplus F$
$V \otimes W$	$\longleftrightarrow$	$E \otimes F$
$\Lambda^i V$		$\Lambda^i E$
$\hom_{\mathbb{R}}(V, W)$		$\hom_{\mathbb{R}}(E, F)$

Canonical isomorphisms yield canonical isom. of vector bundles, e.g. $(V_1 \oplus V_2) \otimes_{\mathbb{R}} V_3 \cong (V_1 \otimes_{\mathbb{R}} V_3) \oplus (V_2 \otimes_{\mathbb{R}} V_3)$.

Def.: An n-dimensional pre-vector bundle on a space B consists of the following data:

- a set E with a surjective map $E \xrightarrow{p} B$
- every fiber $E_b = p^{-1}(b)$ is endowed with the structure of an n-dim. vector space
- an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of B
- for every $i \in I$, a map

$$\begin{tikzcd}
p^{-1}(U_i) \arrow[r, "h_i"] \arrow[d] & U_i \times \mathbb{R}^n \arrow[d] \\
U_i \arrow[r, "="] & U_i
\end{tikzcd}$$

s.th. h_i is a linear isomorphism in each fiber ↺

- For $i,j\in I$ the function

$$\begin{aligned} U_i \cap U_j &\longrightarrow GL(n,\mathbb{R}) \\ x &\longmapsto (h_j)_x \circ (h_i)_x^{-1} \end{aligned}$$

is continuous.

Prop.: In the setting above, there is exactly one topology on E s.th. $E \xrightarrow{p} B$ is a vector bundle for which the maps h_i are local trivialisations.

Proof: Let $\mathcal{O}$ be the topology on E generated by

$$h_i^{-1}(V)\ ,\ i\in I,\ V\subset U_i\times\mathbb{R}^n \text{ open}.$$

This is the unique topology on E for which $p^{-1}(U_i)\subset E$ is open and h_i is a homeomorphism.
We use here that $\mathcal{O}|_{p^{-1}(U_i)}$ is generated by $h_i^{-1}(U)$, $V\subset U_i\times\mathbb{R}^n$, which is implied by the last condition. ▨

How do we apply this to the standard constructions?
Let

$$\begin{tikzcd} E \arrow[d, "p_E"] & F \arrow[d, "p_F"] \\ B & B \end{tikzcd}$$

be two vector bundles.

We define the structure of a pre-vector bundle on

$$\begin{tikzcd} E\otimes F := \bigcup_{b\in B} E_b\otimes F_b \arrow[d] \\ B \end{tikzcd}$$

Choose an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of B s.th.

E_{U_i} and F_{U_i} are trivial with respect to
$$\begin{tikzcd} E_{U_i} \arrow[r, "\cong", "\varphi_i"'] \arrow[d] & U_i \times \mathbb{R}^n \arrow[d] \\ U_i \arrow[r, equal] & U_i \end{tikzcd},$$

and
$$\begin{tikzcd} F_{U_i} \arrow[r, "\cong", "\psi_i"'] \arrow[d] & U_i \times \mathbb{R}^n \arrow[d] \\ U_i \arrow[r, equal] & U_i \end{tikzcd}.$$

Now set
$$\begin{tikzcd} (E \otimes F)_{|U_i} = \bigcup\limits_{x \in U_i} E_x \otimes F_x \arrow[r, "h_i", "\cong"'] \arrow[d] & U_i \times \overset{\mathbb{R}^{mn}}{\overset{\wr}{(\mathbb{R}^m \underset{\mathbb{R}}{\otimes} \mathbb{R}^n)}} \arrow[d] \\ U_i \arrow[r, "="] & U_i \end{tikzcd}$$

with $h_i(\underset{\underset{E_x \otimes F_x}{\rotatebox{90}{$\in$}}}{e \otimes f}) = (x, \; pr_2 \circ \varphi_i(e) \otimes pr_2 \circ \psi_i(f))$. The maps

$$\begin{aligned} U_i \cap U_j &\longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^m \underset{\mathbb{R}}{\otimes} \mathbb{R}^n) \cong GL(mn, \mathbb{R}) \\ \rotatebox{90}{$\in$} \quad & \\ x &\longmapsto (\varphi_j)_x \circ (\varphi_i)_x^{-1} \otimes (\psi_j)_x \circ (\psi_i)_x^{-1} \end{aligned}$$

for $i, j \in I$ are indeed continuous.

Def.: A section of a vector bundle $E \xrightarrow{p} B$ is a continuous map $s : B \longrightarrow E$ such that $p \circ s = id_B$.

Def.: A Riemannian metric on a vector bundle $E \xrightarrow{p} B$ is a section of $(E \otimes E)^*$ s.th. $s(x) : E_x \underset{\mathbb{R}}{\otimes} E_x \longrightarrow \mathbb{R}$ is a scalar product for each $x \in B$.

↺

Choose an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of B s.th.

E_{U_i} and F_{U_i} are ~~respect~~ trivial with respect to

$$\begin{tikzcd} E_{U_i} \arrow[r, "\cong", "\varphi_i"'] \arrow[d] & U_i \times \mathbb{R}^n \arrow[d] \\ U_i \arrow[r, equal] & U_i \end{tikzcd},$$

and

$$\begin{tikzcd} F_{U_i} \arrow[r, "\cong", "\psi_i"'] \arrow[d] & U_i \times \mathbb{R}^n \arrow[d] \\ U_i \arrow[r, equal] & U_i \end{tikzcd}.$$

Now set

$$\begin{tikzcd} (E \otimes F)_{|U_i} = \bigcup\limits_{x \in U_i} E_x \otimes F_x \arrow[r, "h_i", "\cong"'] \arrow[d] & U_i \times (\overset{\overset{\mathbb{R}^{mn}}{\shortparallel}}{\mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{R}^n}) \arrow[d] \\ U_i \arrow[r, "="] & U_i \end{tikzcd}$$

with $h_i(\underset{\underset{E_x \otimes F_x}{\cap}}{e \otimes f}) = (x, \ pr_2 \circ \varphi_i(e) \otimes pr_2 \circ \psi_i(f))$. The maps

$$\begin{aligned} U_i \cap U_j &\longrightarrow Aut_{\mathbb{R}}(\mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{R}^n) \cong GL(mn, \mathbb{R}) \\ \overset{\cup\!\shortmid}{x} &\longmapsto (\varphi_j)_x \circ (\varphi_i)_x^{-1} \otimes (\psi_j)_x \circ (\psi_i)_x^{-1} \end{aligned}$$

for $i, j \in I$ are indeed continuous.

Def.: A <u>section</u> of a vector bundle $E \xrightarrow{p} B$ is a continuous map $s : B \longrightarrow E$ such that $p \circ s = id_B$.

Def.: A <u>Riemannian metric</u> on a vector bundle $E \xrightarrow{p} B$ is a section of $(E \otimes E)^*$ s.th. $s(x) : E_x \otimes_{\mathbb{R}} E_x \longrightarrow \mathbb{R}$ is a scalar product for each $x \in B$.

↺

21.12.

Def.: A family of functions

$$T = \left(\tau_i : X \to \mathbb{R}_{\geq 0} \right)_{i \in I} \quad \text{on a space } X$$

s.th. every $x \in X$ has a neighborhood that meets only finitely many $\mathrm{supp}(\tau_i)$ (we say T is <u>locally finite</u>) and for every $x \in X$

$$\sum_{i \in I} \tau_i(x) = 1$$

is called a <u>partition of unity</u> on X.

If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X, then a partition of unity $T = \{\tau_i\}_{i \in I}$ with $\mathrm{supp}(\tau_i) \subset U_i$ is called <u>subordinate</u> to $\mathcal{U}$.

Further, an open cover $\mathcal{U}$ is called <u>numerable</u> if there is a <u>partition of unity</u> subordinate to $\mathcal{U}$.

Recall that an open cover $\mathcal{V} = \{V_j\}_{j \in J}$ is a <u>refinement</u> of an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ if for every $j \in J$ there is $i \in I$ with $V_j \subset U_i$.

Def.: A Hausdorff space is <u>paracompact</u> if every open cover has a numerable refinement.

Thm.: (without proof)

Metrizable spaces, CW-complexes and manifolds are paracompact. Further, compact Hausdorff spaces are paracompact.

Thm.: A vector bundle over a paracompact space possesses a Riemannian metric.

Proof: A trivial vector bundle has a Riemannian metric.

Let $E \to B$ be an arbitrary vector bundle, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of B s.th. $E|_{U_i}$ are trivial. Upon taking a refinement, we may assume that $\mathcal{U}$ is numerable, so there is a partition of unity $\{\tau_i\}_{i\in I}$ with $\operatorname{supp}(\tau_i) \subset U_i$. Let s_i be a Riemannian metric of E_{U_i}. Define for $x \in X$:

$$s(x) = \sum_{i\in I} \tau_i(x)\cdot s_i(x) \in (E_x \underset{\mathbb{R}}{\otimes} E_x)^*$$

This defines a continuous section s ("$0 \cdot$ undefined $= 0$") of the vector bundle $(E\otimes E)^*$. Further, s is a Riemannian metric since the subspace of scalar products within the vector space of bilinear forms is convex. ▨

Cor.: Let E be a <u>real</u> vector bundle.
Then $E^* \cong E$.

Proof: Choose a Riemannian metric s on E.
Then $E \to E^*$, $E_x \ni v \longmapsto \big(w \longmapsto \underset{\overset{\|}{\langle v,w\rangle}}{s(x)(v\otimes w)}\big)$
is an isomorphism. ▨

BTW: this fails for complex vector bundles. Reason: A hermitian scalar product is not $\mathbb{C}$-linear in both variables.

8.3 The tangent bundle

Let $M \subseteq \mathbb{R}^n$ be a k-dim. smooth manifold.

Recall that we defined the tangent space at $p \in M$ as

$$T_pM = \left\{ X \in \mathbb{R}^n \;\middle|\; X = \dot{\alpha}(0) \text{ for a smooth curve } \alpha : (-\varepsilon,\varepsilon) \to M \text{ with } \alpha(0)=p \right\}$$

$$= \left\{ \alpha : (-\varepsilon,\varepsilon) \to M \text{ smooth} \;\middle|\; \alpha(0)=p \right\} \big/ \sim$$

with $\alpha \sim \beta \iff \dot{\alpha}(0) = \dot{\beta}(0)$.

How can we define $\sim$ intrinsically – without referring to the ambient $\mathbb{R}^n$?

$\dot{\alpha}(0) = \dot{\beta}(0) \iff$ for every smooth $\mathbb{R}$-valued function f defined in a neighborhood of p in M we have

$$\frac{d}{dt}\Big|_{t=0} (f \circ \alpha) = \frac{d}{dt}\Big|_{t=0} (f \circ \beta) .$$

"$\Rightarrow$": Chain rule

"$\Leftarrow$": Suppose $\dot{\alpha}(0) \neq \dot{\beta}(0)$, say the i-th components differ.
Let f be the projection to the i-th component.
Then $\frac{d}{dt}\Big|_{t=0} (f \circ \alpha) \neq \frac{d}{dt}\Big|_{t=0} (f \circ \beta)$.

Let

$$\mathcal{E}_p(M) := \left\{ \overset{\substack{M \\ \cup\!|}}{\underset{\substack{\cup \\ p}}{U}} \xrightarrow{\;f\;} \mathbb{R} \text{ smooth} \right\} \big/ \sim \quad \text{with}$$

$$f \sim g \iff f = g \text{ on some neighborhood of } p.$$

We call $[f]$ the <u>germ</u> of f at p.

So we can characterize T_pM as

$$T_pM = \left\{ (-\varepsilon,\varepsilon) \xrightarrow{\alpha} M \;\middle|\; \alpha \text{ smooth},\ \alpha(0)=p \right\} \big/ \sim$$

↗

with $\alpha \sim \beta \Longleftarrow \frac{d}{dt}\Big|_{t=0}(f\circ\alpha) = \frac{d}{dt}\Big|_{t=0}(f\circ\beta)$ for all $[f] \in \mathcal{E}_p(M)$.

Only the functionals $\mathcal{E}_p(M) \longrightarrow \mathbb{R}$, $[f] \longmapsto \frac{d}{dt}\Big|_{t=0}(f\circ\alpha)$

are relevant here!

Def.: Let M be a smooth manifold and $p \in M$.

A derivation on $\mathcal{E}_p(M)$ is a $\mathbb{R}$-linear map

$$D: \mathcal{E}_p(M) \longrightarrow \mathbb{R}$$

s.th. $D([f]\cdot[g]) = D([f])\cdot g(p) + f(p)\cdot D([g])$

for all $[f],[g] \in \mathcal{E}_p(M)$.

Define the tangent space T_pM of M at p as the $\mathbb{R}$-vector space of derivations on $\mathcal{E}_p(M)$. (with pointwise addition and scalar product)

Elements in T_pM are called tangent vectors at p.

Let $h: U \longrightarrow \mathbb{R}^n$ be a (smooth) chart around $p \in M$ of the n-dim. smooth manifold M. We obtain derivations

$$\partial_i : \mathcal{E}_p(M) \longrightarrow \mathbb{R}, \quad f \longmapsto \frac{\partial}{\partial x_i}\Big|_{x=h(p)}(f\circ h^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}).$$

Prop.: $\{\partial_i\}_{i=1,\dots,n}$ is a basis of T_pM.

Lemma: Let f be a smooth $\mathbb{R}$-valued function on some neighborhood U of $0 \in \mathbb{R}^n$. There are smooth functions

$$f_i : U \longrightarrow \mathbb{R}, \quad i \in \{1,\dots,n\}$$

s.th. $f(x) = f(0) + \sum_{i=1}^{n} x_i \cdot f_i(x)$ for $x \in U$.

↻

Proof: Set $f_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n)\, dt$. ▨

Proof of the proposition:

We may assume $h(p) = 0 \in \mathbb{R}^n$.

Let $a_1, \dots, a_n \in \mathbb{R}$ be s.th. $a_1 \partial_1 + \dots + a_n \partial_n = 0$.

Consider $f_j : U \xrightarrow{h} \mathbb{R}^n \xrightarrow{pr_j} \mathbb{R}$.

Then
$$0 = \left(\sum_{j=1}^{n} a_i \partial_i\right)([f_j]) = \sum_{j=1}^{n} a_i \underbrace{\frac{\partial}{\partial x_i}\Big|_{x=0}(pr_j)}_{=\delta_{ij}} = a_j .$$

→ linear independance

Let $D : \mathcal{E}_p(M) \longrightarrow \mathbb{R}$ be a derivation. Let $a_i = D(f_i)$.

Want to show:
$$\overline{D} := D - \sum_{i=1}^{n} a_i \partial_i = 0$$

Clearly, $\overline{D}(f_j) = 0$ for $j \in \{1, \dots, n\}$.

For $[f] \in \mathcal{E}_p(M)$ we have g_i's as in the previous lemma so that
$$f \circ h^{-1}(x) = \sum_{i=1}^{n} x_i \cdot g_i(x) + f(p) .$$

Equivalently,
$$f(y) = \sum_{i=1}^{n} \underbrace{pr_i(h(y))}_{= f_i(y)} (g_i \circ h)(y) + f(p) \qquad \text{for } y \in U.$$

So $[f] = \sum_{i=1}^{n} (pr_i \circ h) \circ (g_i \circ h) + \underbrace{f(p)}_{= \text{const.}} \in \mathcal{E}_p(M)$.

Now $\overline{D}([f]) = \sum_{i=1}^{n} \underbrace{\overline{D}(f_i)}_{=0} \cdot g_i(0) + \underbrace{f_i(0)}_{=0} \cdot \overline{D}(g_i \circ h) = 0$. ▨

↗

Def.: Let $f: M \longrightarrow N$ be a smooth map between smooth manifolds. Let
$$\begin{aligned} \mathcal{E}_p(f): \mathcal{E}_{f(p)}(N) &\longrightarrow \mathcal{E}_p(M) \\ [g] &\longmapsto [g \circ f] \end{aligned}.$$

The linear map
$$\begin{aligned} T_pf: T_p M &\longrightarrow T_{f(p)} N \\ \rotatebox{90}{$\in$} \quad & \quad\;\; \rotatebox{90}{$\in$} \\ D &\longmapsto D \circ \mathcal{E}_p(f) \end{aligned}$$
is called the

differential of f at p.

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We see immediately that

- $T_p(f \circ g) = T_{g(p)} f \circ T_p g$
- $T_p(id_M) = id_{T_pM}$

In particular, (local) diffeo's induce isomorphisms on tangent spaces.

Prop.: Let $f: M \longrightarrow N$ be smooth. Choose smooth charts $h: U \xrightarrow{\cong} \mathbb{R}^m$ and $k: V \xrightarrow{\cong} \mathbb{R}^n$ with $f(U) \subset V$. Let $p \in U$.

The differential $T_p f$ is represented by the Jacobi matrix of $k \circ f \circ h^{-1}$ at $h(p)$ with respect to bases $\{\partial_i^h\}$ of T_pM and $\{\partial_j^k\}$ of $T_{f(p)}N$ associated to the charts h and k.

Proof: To show:
$$T_pf\,(\partial_i^h) = \sum_{j=1}^{n} \frac{\partial(pr_j \circ k \circ f \circ h^{-1})}{\partial x_j}\bigg|_{x=h(p)} \cdot \partial_j^k \in T_{f(p)} N$$

Let $[g] \in \mathcal{E}_{f(p)}(N)$. Then $T_pf\,(\partial_i^h)([g]) = \partial_i^h([g \circ f]) = \dfrac{\partial(g \circ f \circ h^{-1})}{\partial x_i}\bigg|_{x=h(p)}$
$$= \frac{\partial((g \circ k^{-1}) \circ (k \circ f \circ h^{-1}))}{\partial x_i}\bigg|_{x=h(p)} \overset{\text{chain}}{\underset{\text{rule}}{=}} \sum_{j=1}^{n} \overbrace{\frac{\partial(g \circ k^{-1})}{\partial x_j}\bigg|_{x=k(f(p))}}^{=\,\partial_j^k[g]} \cdot \frac{\partial(pr_j \circ k \circ f \circ h^{-1})}{\partial x_j}\bigg|_{x=h(p)}.$$ ▨

Rem.: The tangent space $T_p\mathbb{R}^n$ of $\mathbb{R}^n$ is canonically identified with $\mathbb{R}^n$ via

$$\begin{aligned} \mathbb{R}^n &\xrightarrow[c_p]{\cong} T_p\mathbb{R}^n \\ e_i &\longmapsto \left([f] \longmapsto \frac{\partial f}{\partial x_i}\Big|_{x=p} \right). \end{aligned}$$

Prop. and Def.: Let M be a smooth manifold. Define

$$\begin{cases} TM = \bigsqcup_{x \in M} T_x M \\ p : TM \longrightarrow M \\ T_xM \ni v \longmapsto x . \end{cases}$$

The fibers $p^{-1}(x) = T_xM$ are endowed with a vector space structure.

Let $\mathcal{A}_{max}$ be the maximal smooth atlas of M:

$$\mathcal{A}_{max} = \{ h_i : U_i \longrightarrow \mathbb{R}^m \mid i \in I \}$$

For $i \in I$ we define

$$\begin{aligned} \overline{h}_i : p^{-1}(U_i) &\longrightarrow U_i \times \mathbb{R}^m \\ T_xM \ni v &\longmapsto \left(x, c_{h_i(x)} \circ T_x h_i(v) \right). \end{aligned}$$

This data defines a prevector bundle, thus a vector bundle, called the Tangent bundle.

Proof: It suffices to prove that

$$\begin{aligned} U_i \cap U_j &\longrightarrow GL_m(\mathbb{R}) \\ \overset{\cup\!\!\!\!\shortmid}{x} &\longmapsto (\overline{h}_j)_x \circ (\overline{h}_i)_x^{-1} \end{aligned}$$

is continuous.

↗

This follows from

$$(\overline{h_j})_x \circ (\overline{h_i})_x^{-1} = c_{h_j(x)} \circ (T_x h_j) \circ (T_x h_i)^{-1} \circ c_{h_i(x)}^{-1}$$

$$= c_{h_j(x)} \circ T_{h_i(x)}(h_j \circ h_i^{-1}) \circ c_{h_i(x)}^{-1}$$

$$= J_{h_i(x)}(h_j \circ h_i^{-1}) .$$

Remark: TM is a smooth $2m$-dim. manifold. A smooth atlas is given by the maps

$$\begin{tikzcd} p^{-1}(U_i) \arrow[d, "\overline{h_i}"] \\ U_i \times \mathbb{R}^m \arrow[d, "h_i \times id"] \\ \mathbb{R}^m \times \mathbb{R}^m \end{tikzcd}$$

Verify that the change of charts is a diffeo!

Examples :

1) $T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m$ trivial

$T_p\mathbb{R}^m \ni v \longmapsto (p, c_p(v))$

2) TS^1 trivial

$T(S^1 \times \ldots \times S^1)$ trivial

$\Big\} \to$ Exercise.

3) TS^2 ?

$j : S^2 \hookrightarrow \mathbb{R}^3$ induces $Tj : TS^2 \longrightarrow j^* T\mathbb{R}^3 \cong S^2 \times \mathbb{R}^3$

$$\begin{tikzcd} TS^2 \arrow[rr] \arrow[dr] & & j^* T\mathbb{R}^3 \cong S^2 \times \mathbb{R}^3 \arrow[dl] \\ & S^2 & \end{tikzcd}$$

What is the image of $TS^2 \longrightarrow j^* T\mathbb{R}^3$?

Consider $f : \mathbb{R}^3 \longrightarrow R$, $x \longmapsto \|x\|^2$. Then $S^2 = f^{-1}(\{1\})$,

So $S^2 \xhookrightarrow{j} \mathbb{R}^3 \xrightarrow{f} R$ is constant, hence has zero differential.

$$\operatorname{im}(Tj) \subset \ker(Tf) = \bigsqcup_{x \in \mathbb{R}^3} \underbrace{\{ v \in \mathbb{R}^3 \mid \langle x, v \rangle = 0 \}}_{= \ker(T_x f)}$$

With $\dim(\operatorname{im} T_x j) = \dim T_x S^2 = 2$ we conclude that

$$TS^2 \cong \{ (x, v) \mid x \in S^2, v \in \mathbb{R}^3, \langle x, v \rangle = 0 \} .$$

↺

TS^2 is not trivial, there is no nowhere vanishing section ($\to$ AT 1).

Let's determine the sphere bundle

$$S(TS^2) = \{ v \in TS^2 \mid s(v,v) = 1 \}$$

with respect to a Riemannian metrics.

we take the standard Riem. metric on $S^2 \times \mathbb{R}^3$ and restrict to $TS^2 \subset S^2 \times \mathbb{R}^3$.

$$\begin{tikzcd} S(TS^2) \arrow[d, "p"] \\ S^2 \cong \mathbb{C}P^1 \end{tikzcd}$$

is a fiber bundle with fiber S^1.

Note that we have a S^1-fiber bundle

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[r, "\eta"] \arrow[d, phantom, "\cap"] & \mathbb{C}P^1 \arrow[d, phantom, "\cong"] \\ & \mathbb{C}^2 & S^3/S^1 \\ & (z_0, z_1) \arrow[r, mapsto] & {[z_0 : z_1]} \end{tikzcd}$$

,

BTW: this is the famous Hopf map.

but $S(TS^2) \not\cong S^3$.

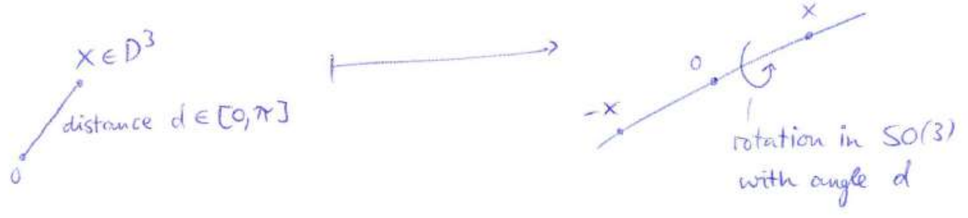
In fact, we have $S(TS^2) \xrightarrow{\cong} SO(3)$ and $SO(3) \cong \mathbb{R}P^3$.

$$S(TS^2) = \left\{ (x,v) \,\middle|\, \begin{array}{l} x, v \in \mathbb{R}^3, \\ \|x\| = \|v\| = 1 \\ \langle x, v \rangle = 0 \end{array} \right\} \longmapsto (x \,|\, v \,|\, x \times v)$$

identify antipodal points on $\partial D^3 = S^2$

The latter homeomorphism is

$$\mathbb{R}P^3 = \underset{\text{ball of radius } \pi}{D^3} / \sim \;\longrightarrow\; SO(3)$$



↗

Example: $T\mathbb{R}P^n$

Claim: $T\mathbb{R}P^n \cong \mathrm{hom}(L, L^{\perp})$, where $L \to \mathbb{R}P^n$ is the tautological line bundle.

$$L = \{([x], v) \mid v \in \mathrm{Span}(x)\} \subseteq \mathbb{R}P^n \times \mathbb{R}^{n+1}$$

$$L^{\perp} = \{([x], w) \mid w \in \mathrm{Span}(x)^{\perp}\} \subseteq \mathbb{R}P^n \times \mathbb{R}^{n+1}$$

Consequence: $T\mathbb{R}P^n \oplus \underset{\text{trivial line bundle}}{\underline{\mathbb{R}}} \cong L^{n+1}$

$$\begin{aligned} T\mathbb{R}P^n \oplus \underline{\mathbb{R}} &\cong \mathrm{hom}(L, L^{\perp}) \oplus \overbrace{\mathrm{hom}(L, L)}^{\text{trivial, since } x \mapsto id_{L_x} \text{ is a non-vanishing section}} \\ &\cong \mathrm{hom}(L, \underbrace{L \oplus L^{\perp}}_{\cong \underline{\mathbb{R}^{n+1}}}) \\ &\cong L^* \oplus \overset{(n+1)}{\cdots} \oplus L^* \\ &\cong L^{n+1}. \end{aligned}$$

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It remains to show that $T\mathbb{RP}^n \cong \hom(L, L^{\perp})$, where $L \to \mathbb{RP}^n$ is the tautological line bundle.

Let $pr: S^n \to \mathbb{RP}^n$ be the 2-fold covering, pr is a local diffeomorphism, thus $TS^n \to pr^* T(\mathbb{RP}^n)$ (induced by Tpr) is an isomorphism of vector bundles.

$$pr^* T\mathbb{RP}^n \cong TS^n \xrightarrow{\varphi} pr^* \hom(L, L^{\perp})$$

$$(x, v) \longmapsto \begin{pmatrix} L_{[x]} \to L^{\perp}_{[x]} \\ \lambda x \mapsto \lambda v \end{pmatrix}$$

φ is a bundle isomorphism, too.

Now φ descends to a bundle isomorphism $T\mathbb{RP}^n \to \hom(L, L^{\perp})$ which is fiberwise given by

$$\begin{tikzcd} T_{[x]}\mathbb{RP}^n & T_x S^n \arrow[l, "\cong"'] \arrow[r, "\varphi_x", "\cong"'] & \hom(L, L^{\perp}) \arrow[d, phantom, "\shortparallel"] \\ & & (pr^* \hom(L, L^{\perp}))_x \end{tikzcd}$$

at $[x] \in \mathbb{RP}^n$.

This is well-defined, i.e. independant of the choice of representative x of $[x]$, because of the commutativity of

$$\begin{tikzcd} T_x S^n \arrow[rr, "T_x\alpha"] \arrow[dr, "T_x pr"'] & & T_x S^n \arrow[dl, "T_{-x} pr"] \\ & T_{[x]}\mathbb{RP}^n & \end{tikzcd}$$

and the fact that $\underset{\substack{\uparrow \\ \text{antipodal map}}}{T_x \alpha} (x, v) = (-x, -v)$,

such that $\varphi(T\alpha(x, v)) = \varphi(-x, -v) = (\lambda(-x) \mapsto \lambda(-v))$

$= (\lambda(x) \mapsto \lambda(v))$

$= \varphi(x, v)$.

↺

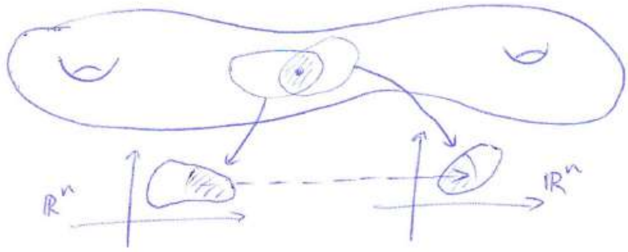
9. More on manifolds

9.1 Orientation of manifolds

Two bases $(b_1, \dots, b_n)$ and $(c_1, \dots, c_n)$ of an n-dimensional vector space are equivalent if $\det(f) > 0$ for the automorphism with $f(b_i) = c_i$ for $i = 1, \dots, n$.
The choice of one of the two equivalence classes is called an orientation.
After fixing an orientation $\mathcal{O}$, a basis is positively oriented if it belongs to $\mathcal{O}$.

Remark: Consider a short exact sequence $0 \to V_1 \xrightarrow{j} V_2 \xrightarrow{p} V_3 \to 0$ of vector spaces. Let $\mathcal{O}_{V_1}$ and $\mathcal{O}_{V_3}$ be orientations of V_1 and V_3. Let $(b_1, \dots, b_n)$ and $(c_1, \dots, c_m)$ be bases of V_1 and V_3 which are positively oriented with respect to $\mathcal{O}_{V_1}$ and $\mathcal{O}_{V_3}$. Choose $d_i \in V_2$ with $p(d_i) = c_i$. Then $(j(b_1), \dots, j(b_n), d_1, \dots, d_m)$ is a basis of V_2 and determines an orientation of V_2, called the orientation induced by $\mathcal{O}_{V_1}$ and $\mathcal{O}_{V_3}$.

Let M be a smooth manifold (with or without boundary).
We call two (smooth) charts of M positively related if the Jacobian of the coordinate change has always positive determinant.



A (smooth) atlas is orienting if any two of its charts are positively related. We call M orientable if M has an orienting atlas.

An orientation of M is represented by an orienting atlas and any two of such define the same orientation if their union contains charts which are positively related.

If M is oriented by an orienting atlas, we call a chart oriented if it is positively related to all charts of the orienting atlas.

The definitions above apply to manifolds of positive dimension.

An orientation of a 0-dimensional manifold is a function $M \to \{\pm 1\}$.

Remark: A smooth connected manifold has at most two different orientations: If an orientation $\mathcal{O}$ is fixed, then every other orientation $\mathcal{O}'$ yields a continuous map $\varepsilon : M \to \{\pm 1\}$, where $\varepsilon(x) = \pm 1$ according to whether a chart around x in $\mathcal{O}'$ is positively related to a chart around $x \in \mathcal{O}$. If M is connected, then ε is constant, proving the claim.

Let M be an oriented smooth manifold. Then there is an induced orientation on each tangent space $T_x M$ by requiring that an oriented chart $h : U \to \mathbb{R}^n$ maps the orientation on $T_x M$ to the standard orientation of $T_{h(x)} \mathbb{R}^n \cong \mathbb{R}^n$.

↺

Def.: Let $E \xrightarrow{p} B$ be an n-dimensional vector bundle. An orientation of $E \xrightarrow{p} B$ consists of an orientation of each fiber E_x which is locally trivial in the following sense: Around each ~~chart~~ point $b \in B$ there is a bundle chart

$$h: E_u = p^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$

which maps the orientation of E_x, $x \in U$, to the standard orientation of $\mathbb{R}^n$. A vector bundle is orientable if it permits an orientation.

The discussion above shows that an orientation of M yields an orientation of TM. This goes both ways.

Suppose an orientation on TM is given.

Then let $\mathcal{A}$ consist of all charts $h: U \to \mathbb{R}^n$ such that $T_x h$ transports the given orientation on $T_x M$, $x \in U$ to the standard orientation of $T_{h(x)} \mathbb{R}^n \cong \mathbb{R}^n$.

We claim that $\mathcal{A}$ is an oriented atlas.

First of all, $\mathcal{A}$ is an atlas. Let $h: U \to \mathbb{R}^n$ be a chart of M around $x \in U \subseteq M$. By the local triviality of the given family of orientations on tangent spaces we assume (by making $\mathcal{U}$ smaller) that either $T_x h$ is orientation preserving for all $x \in U$ (taking the standard orientation) on $T_{h(x)} \mathbb{R}^n \cong \mathbb{R}^n$ or $T_x h$ is orientation reversing for all $x \in U$.

In the latter case one may post-compose h with $\mathbb{R}^n \to \mathbb{R}^n$, $(x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, +x_n)$ to obtain a chart in $\mathcal{A}$ with domain U.

So $\mathcal{A}$ is an atlas. Now consider two charts in $\mathcal{A}$, $h: U \to \mathbb{R}^n$ and $\ell: V \to \mathbb{R}^n$. Since $\mathbb{R}^n = T_{\ell(x)}\mathbb{R}^n \xleftarrow{T\ell} T_x M \xrightarrow{Th} T_{h(x)}\mathbb{R}^n = \mathbb{R}^n$ is given by the Jacobian of $h \circ \ell^{-1}$ at $\ell(x)$, $x \in U \cap V$, the charts h, ℓ are positively related.

So $\mathcal{A}$ is oriented.

Thm.: An orientation of a smooth manifold is the same as an orientation of its tangent bundle.

Thm.: Let $f: M \to N$ be a smooth map of smooth manifolds and $y \in N$ a regular value. Then $f^{-1}(y)$ is a smooth manifold (even submanifold of M) of dimension $\dim(M) - \dim(N)$. Furthermore, the short exact sequences

$$0 \to T_x f^{-1}(y) \to T_x M \xrightarrow{T_x f} T_y N \to 0$$

for every $x \in f^{-1}(y)$ induce an orientation of $Tf^{-1}(y)$ when orientations of TM and TN are given.

Proof: The first statement is known from EGT.

Further it is clear that $T_x f^{-1}(y) \subseteq \ker T_x f$, and from the dimensions exactness of $(*)$ follows. The orientations of $T_x M$ and $T_y N$ determine an orientation of $T_x f^{-1}(y)$ (cf. previous remark).

Local triviality of orientations is left as an exercise. $\boxtimes$

Examples:

1.) Spheres (as preimages of $f(x_1,\dots,x_n) = x_1^2+\dots+x_n^2$) are orientable.

2.) $\mathbb{R}P^n$ is orientable if n is odd.

Let $\alpha : S^n \to S^n$ be the antipodal map $\alpha(x) = -x$.

$$\begin{tikzcd}
0 \arrow[r] & T_xS^n \arrow[r] \arrow[d, "T_x\alpha"] & T_x\mathbb{R}^{n+1} \arrow[r, "T_xf"] \arrow[d, "T_x\alpha"] & T_x\mathbb{R} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & T_{-x}S^n \arrow[r] & T_{-x}\mathbb{R}^{n+1} \arrow[r, "T_{-x}f"] & T_{-x}\mathbb{R} \arrow[r] & 0
\end{tikzcd}$$

This diagram commutes.

Hence $T_x\alpha : T_xS^n \to T_xS^n$ is orientation preserving if and only if $(-1)^{n+1}\,\mathrm{id}_{\mathbb{R}^{n+1}} = T_x\alpha : T_x\mathbb{R}^{n+1} \to T_x\mathbb{R}^{n+1}$ is orientation preserving, i.e. for odd n.

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We define an orientation of $T\mathbb{R}P^n$ as follows (n odd)

For $y \in \mathbb{R}P^n$ pick a preimage $x \in S^n$ with $p(x) = y$ and take the orientation coming from the isomorphism $T_xS^n \xrightarrow[T_xp]{\cong} T_y\mathbb{R}P^n$.

If one picks the other point $-x$ in $p^{-1}(y)$, one obtains the same orientation because of the commutativity of

(**)

$$\begin{tikzcd}
T_{-x}S^n \arrow[rr, "T_{-x}\alpha"] \arrow[dr, "T_{-x}p"'] & & T_xS^n \arrow[dl, "T_xp"] \\
& T_y\mathbb{R}P^n &
\end{tikzcd}$$

orientation-preserving

The local triviality of this family of orientations on $T_y\mathbb{R}P^n$, $y \in \mathbb{R}P^n$, follows from the fact that p is a local diffeomorphism.

3.) $\mathbb{R}P^n$ is <u>not</u> orientable for even n.

↗

Proof: Suppose $T\mathbb{R}P^n$ possesses an orientation.
Via Tp this defines an orientation of TS^n.
There are two possible orientations of TS^n, by changing the orientation on $T\mathbb{R}P^n$, we may assume that we obtain the same orientation for TS^n as from Example (1).
By $(**)$, T_α is orientation preserving.
But this is only possible for odd n.

9.2 Manifolds with boundary

Next, we extend some notions from manifolds to manifolds with boundary.
A (topological) n-dimensional manifold with boundary is a Hausdorff space M with countable basis such that each point has an open neighbourhood which is homeomorphic to an open subset in $\mathbb{R}^n_- = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \leq 0\} \subseteq \mathbb{R}^n$. Such a homeomorphism is called a chart.
If $A \subseteq \mathbb{R}^n$ is any subset, we call $f: A \to \mathbb{R}^n$ smooth if for each $a \in A$ there is an open neighbourhood $\mathcal{U}$ of A and a smooth extension $F: U \to \mathbb{R}^n$ of f. We only apply this to open subspaces of half spaces $\mathbb{R}^n_\pm$. In that case, the differential of F at $a \in A$ is independant of the extension F.
With that it is straightforward to define the notions of smooth atlas, smooth structure etc. for smooth manifolds with boundary.

↺

Let M be a manifold with boundary. Its boundary ∂M is the following subset of M: A point $x \in M$ is in ∂M if there is a chart $h: \underset{x}{\underset{\cup\!\!\!\!|}{U}} \longrightarrow V \subseteq \mathbb{R}^n_-$ such that $h(x) \in \{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n_-$.

The complement of ∂M in M is called the interior of M.
The independence of the choice of the chart in the definition of ∂M follows from the lemma.

Lemma: Let $f: V \longrightarrow W$ be a homeomorphism between open subsets of $\mathbb{R}^n_-$. Then: $f(V \cap \{0\} \times \mathbb{R}^{n-1}) = W \cap \{0\} \times \mathbb{R}^{n-1}$

Proof: The proof is based on a fundamental fact:

Invariance of domain (Hatcher, Thm. 2B.3)
If an open subset of $\mathbb{R}^n$ is homeomorphic to a subset E of $\mathbb{R}^n$, then E is open.

$V \setminus \{0\} \times \mathbb{R}^{n-1}$ is open in $\mathbb{R}^n$. By invariance of domain, the image $f(V \setminus \{0\} \times \mathbb{R}^{n-1})$ is open in $\mathbb{R}^n$. Hence, $f(V \setminus \{0\} \times \mathbb{R}^{n-1})$ has no common point with $\{0\} \times \mathbb{R}^{n-1}$,

$$f(V \setminus \{0\} \times \mathbb{R}^{n-1}) \cap \{0\} \times \mathbb{R}^{n-1} = \emptyset .$$

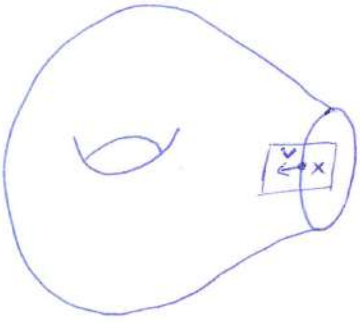
Hence, $f(V \cap \{0\} \times \mathbb{R}^{n-1}) = W \cap \{0\} \times \mathbb{R}^{n-1}$. ▨

↗

Prop.: Let M be an n-dimensional (smooth) manifold with boundary. The boundary ∂M is an $(n-1)$-dimensional (smooth) manifold.

Proof: If $\mathcal{A} = \{ h_i : U_i \longrightarrow V_i \subseteq \mathbb{R}^n_- \}$ is a (smooth) atlas for M, then $\{ h_i|_{U_i \cap \partial M} : U_i \cap \partial M \longrightarrow V_i \cap \{0\} \times \mathbb{R}^{n-1} \}$ is a (smooth) atlas for ∂M. ▨

The notions of smooth maps, tangent spaces / bundles and differentials are defined for smooth manifolds with boundary in the same way.



Let M be a smooth manifold with boundary. Let $x \in \partial M$. We say that $v \in T_x M$ is *pointing inwards* (*outwards*/*tangent*) if there is a chart $h : U \to V \subseteq \mathbb{R}^n_-$ around x s.th. $T_x h : T_x M \longrightarrow T_{h(x)} \mathbb{R}^n_- = \mathbb{R}^n$ maps v to a vector with negative (positive / zero) first coordinate.

Prop.: If a (smooth) manifold M with boundary is orientable, then so is ∂M.

Proof: Let $j : \partial M \hookrightarrow M$ be the inclusion. We have a short exact sequence $0 \longrightarrow T_x \partial M \xrightarrow{T_x j} T_x M \longrightarrow \operatorname{coker}(T_x j) \longrightarrow 0$ for $x \in \partial M$. $\operatorname{coker}(T_x j)$ is oriented so that any outwards pointing vector of $T_x M$ yields a positively oriented basis of $\operatorname{coker}(T_x j)$.

↱

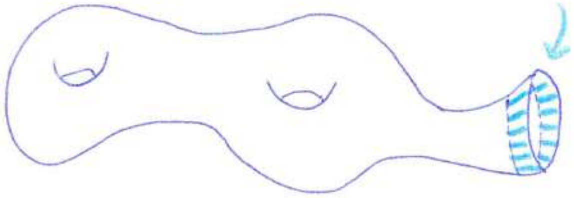
By a previous remark, this yields an orientation of $T_x \partial M$.
("2 out of 3" for orientations)

9.3 Glueing manifolds

Def.: A collar of a smooth n-dimensional manifold with boundary is a diffeomorphism

$$\varphi : [0,1) \times \partial M \longrightarrow U \underset{\text{open}}{\subseteq} M$$

to some open subset of M.



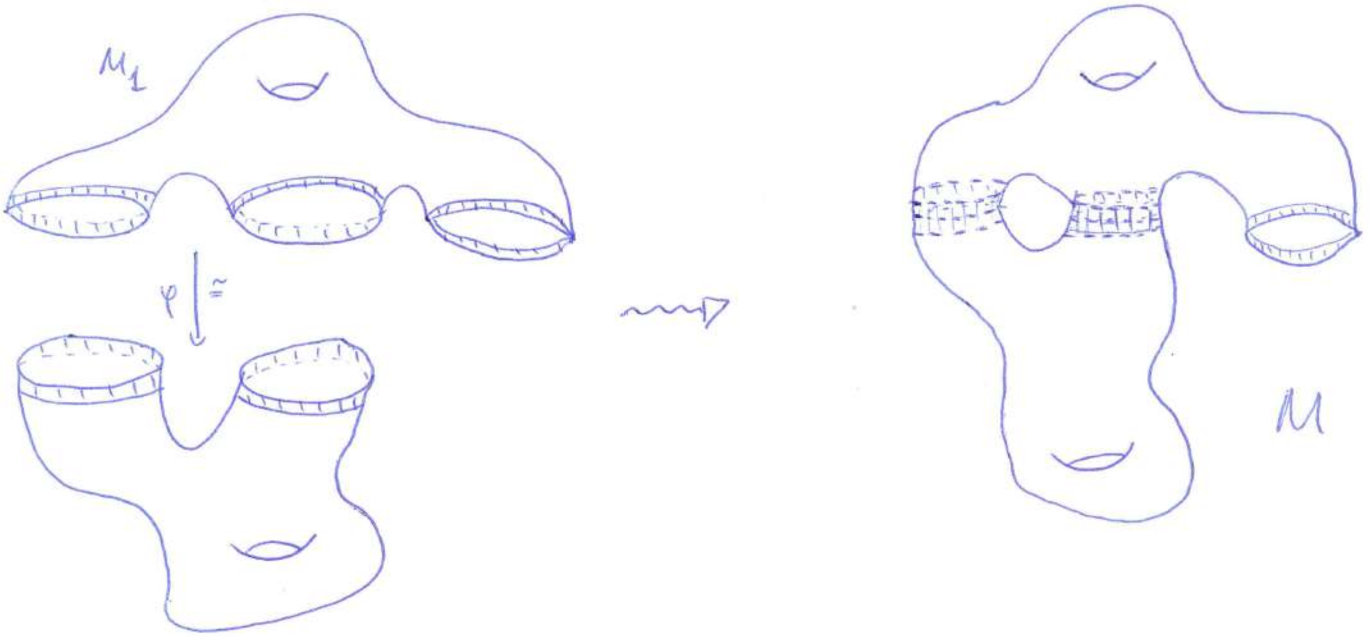
Prop.: Every smooth manifold with boundary has a collar.
(See tom Dieck, Algebraic Topology, Prop. 15.7.8 on p.382)

18.1.16.

Now let M_1 and M_2 be smooth manifolds with boundaries $\partial M_1 \neq \emptyset$, $\partial M_2 \neq \emptyset$. Let N_i be a union of path components of ∂M_i. Let $\varphi : N_1 \to N_2$ and let M be defined by the pushout

$$\begin{tikzcd} N_1 \arrow[r, hook] \arrow[d, hook] & M_2 \arrow[d, hook] \\ M_1 \arrow[r, hook] & M \end{tikzcd}$$

.



We denote the image of M_i in M also by M_i.
Then $M_i \subset M$ is closed and $M_i \setminus N_i$ is open in M.
Next we define the structure of a smooth manifold (possibly with boundary) on M. To this end, we choose collars

$$\psi_i : [0,1) \times \partial M_i \longrightarrow M_i ,$$

with open image $U_i \subset M_i$.
Now define

$$\psi : (-1,1) \times N_1 \longrightarrow M$$

with open image $\mathcal{U}$ by

$$\psi(t,x) = \begin{cases} \psi_1(t,x) , & t \geq 0 , \\ \psi_2(-t, \varphi(x)) , & t < 0 . \end{cases}$$

We declare the smooth structure on M by requiring that

↺

$$M_1 \setminus N_1 \hookrightarrow M$$

$$M_2 \setminus N_2 \hookrightarrow M$$

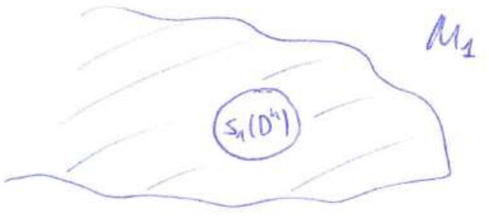
$$(-1,1) \times N_1 \xhookrightarrow{\psi} M$$

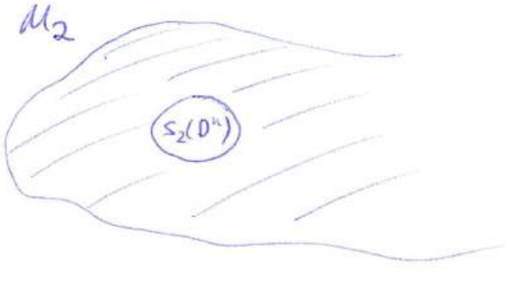
are smooth embeddings. In other words, a smooth atlas for M is given by the union of a smooth atlas $\mathcal{A}_1$ of $M_1 \setminus N_1$, $\mathcal{A}_2$ of $M_2 \setminus N_2$ and a smooth atlas $\mathcal{A}$ coming from the smooth structure on $(-1,1) \times N_1$, i.e.

$$\mathcal{A} = \left\{ h \circ \psi^{-1} \;\middle|\; h : V \to V' \text{ smooth chart of } (-1,1) \times N_1 \right\}.$$

The union is indeed a smooth atlas: The change of coordinates between charts in $\mathcal{A}_i$ and $\mathcal{A}$ is smooth since ψ_i, $i = 1,2$, and φ are diffeomorphisms (onto their image).

Def.: Let M_1 and M_2 be smooth manifolds of the same dimension n. Let $S_i : D^n \hookrightarrow M_i$, $i=1,2$, be a smooth embedding. Then $M_i \setminus s_i(\mathring{D}^n)$ is a smooth manifold with boundary $s_i(S^{n-1}) \cong S^{n-1}$.



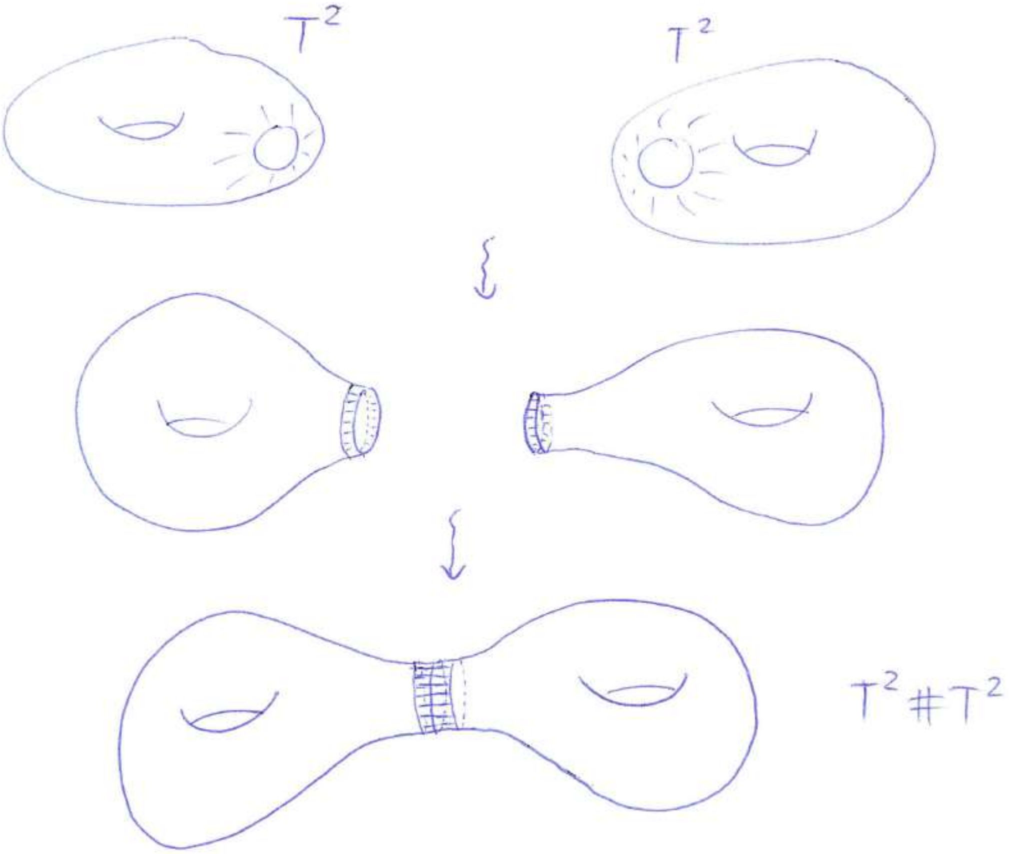


The manifold obtained by glueing $M_1 \setminus s_1(\mathring{D}^n)$ and $M_2 \setminus s_2(\mathring{D}^n)$ along their boundaries via the diffeomorphism

$$s_2 \circ s_1^{-1} : s_1(S^{n-1}) \longrightarrow s_2(S^{n-1})$$

is called the <u>connected sum</u> of M_1 and M_2 and is denoted by $M_1 \# M_2$.

1.)



2.) $M \# S^n \cong M$

<u>Remark</u> (Classification of surfaces):

Every orientable closed surface is a connected sum of tori or S^2.
Every non-orientable closed surface is a connected sum of $\mathbb{R}P^2$'s.

The concatenation of surface words corresponds to the connected sum.

↺

Remark: If M_1 and M_2 are oriented and s_1 is orientation preserving and s_2 is orientation reversing, then $M_1 \# M_2$ can be oriented as follows: On $M_i \setminus s_i(S^{n-1})$ we take the orientation from M_i and on the image of

$$\psi : (-1,1) \times s_1(S^{n-1}) \longrightarrow M_1 \# M_2$$

we transport the orientation of $(-1,1) \times s_1(S^{n-1})$.

This is well-defined since

$$\widetilde{\psi}_1 : [0,1) \times s_1(S^{n-1}) \xrightarrow{\psi_1} M_1 \setminus s_1(S^{n-1}) \hookrightarrow M_1 \# M_2$$

and

$$\widetilde{\psi}_2 : (-1,0] \times s_1(S^{n-1}) \xrightarrow[(t,x)\mapsto(-t,x)]{} \overset{(\text{reversing})}{} [0,1) \times s_1(S^{n-1}) \to$$

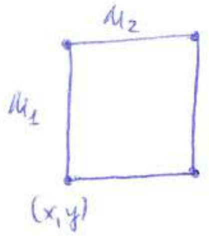
$$\xrightarrow[(\text{reversing})]{id \times (s_2 \circ s_1^{-1})} [0,1] \times s_2(S^{n-1}) \xrightarrow{\psi_2} M_2 \setminus s_2(S^{n-1})$$

are orientation-preserving.

Remark: If the M_i are connected, then the (orientable) diffeomorphism type of $M_1 \# M_2$ does not depend on the choice of the s_i's.
(see Bröcker - Jänich : Differential Topology)

Next we consider smooth manifolds M_1 and M_2 with boundaries.

Problem: How should a chart around (x,y) look like?



$M_1 = [0,1] = M_2$

$M_1 \times M_2 = [0,1] \times [0,1]$

We define a smooth structure on $M_1 \times M_2 \setminus (\partial M_1 \times \partial M_2)$ by taking products of charts of M_1 and M_2.

To deal with the "corner points" $\partial M_1 \times \partial M_2$, we choose collars

$$\Psi_i : \mathbb{R}_- \times \partial M_i \longrightarrow M_i$$

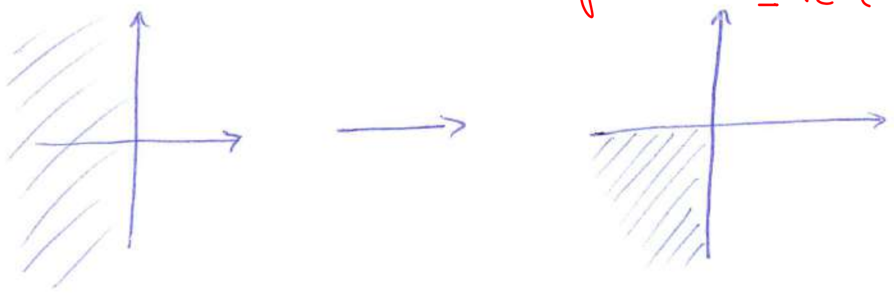
and consider the diagram

$$\begin{tikzcd}
\mathbb{R}^2_- \times \partial M_1 \times \partial M_2 \arrow[r, "\pi \times id", "\cong"'] \arrow[d, "\lambda"] & \mathbb{R}_- \times \mathbb{R}_- \times \partial M_1 \times \partial M_2 \arrow[d, "12"', "\text{interchange factors}"] \\
M_1 \times M_2 & (\mathbb{R}_- \times \partial M_1) \times (\mathbb{R}_- \times \partial M_2) \arrow[l, "\Psi_1 \times \Psi_2"']
\end{tikzcd}$$

whose commutativity defines λ. Here π is the map

π is a diffeomorphism $\mathbb{R}^2_- \setminus \{0\} \to \mathbb{R}_- \times \mathbb{R}_- \setminus \{0\}$

(*)



given by $(r, \varphi) \longmapsto \left(r, \frac{1}{2}\varphi + \frac{3}{2}\pi\right)$ in polar coordinates.

There exists a unique smooth structure on $M_1 \times M_2$ such that

$$M_1 \times M_2 \setminus (\partial M_1 \times \partial M_2) \hookrightarrow M_1 \times M_2$$

and

$$\lambda : \mathbb{R}^2_- \times \partial M_1 \times \partial M_2 \hookrightarrow M_1 \times M_2$$

are smooth embeddings. You need that both smooth structures coincide on their intersection. This follows from (*).

We say that this smooth structure is obtained by smoothening the corners.

10. Duality

10.1 Homological orientation

In the sequel, let R be a commutative ring.

Lemma: Let M be a manifold of Dimension d.

Then

$$H_i(M, M\setminus\{x\}; R) \cong \begin{cases} R & , \text{ if } i=d, \\ 0 & , \text{ else}, \end{cases}$$

for every $x \in M$.

Proof: Let U be the domain of a chart around x. Then:

$$(U, U\setminus\{x\}) \cong (\mathbb{R}^d, \mathbb{R}^d\setminus\{0\})$$

By excision,

$$\begin{tikzcd} H_i(M, M\setminus\{x\}; R) & H_i(U, U\setminus\{x\}; R) \arrow[l, "\cong"'] \arrow[d, phantom, "\wr\|"] \\ & H_i(\mathbb{R}^d, \mathbb{R}^d\setminus\{0\}; R). \end{tikzcd}$$

▨

Def.: Let M be a d-dimensional Manifold. An R-orientation of M consists of a family of generators $\mu_x \in H_d(M, M\setminus\{x\}; R)$, $x \in M$, s.th. (μ_x) satisfies the local triviality condition: ↗

Around each $x \in M$ there is a neighborhood U of x and an element $\mu_U \in H_d(M, M \setminus U; R)$ that maps to μ_y by the map induced by $(M, M\setminus U) \to (M, M\setminus\{y\})$ for all $y \in U$.

19.1.16

μ_x is called a local R-orientation at x.

If M has an R-orientation, it is R-orientable or R-oriented provided an R-orientation is fixed.

In the case $R = \mathbb{Z}$, we drop "R-".

Theorem: A smooth manifold is orientable if and only if it is orientable in the homological sense above.

Proof: Assume M has an orienting atlas $\mathcal{A}$.

Fix a generator $v \in H_d(\mathbb{R}^d, \mathbb{R}^d - \{0\}) \cong \mathbb{Z}$, where $d = \dim(M)$.

Pick a positively oriented chart $h: U \longrightarrow \mathbb{R}^d$ around $x \in M$ with $h(x) = 0$.

Let μ_x be the image of v under

$$H_d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \xrightarrow[\cong]{H_d(h^{-1})} H_d(U, U\setminus\{x\}) \xrightarrow[\cong]{H_d(j)} H_d(M, M\setminus\{x\}) .$$

This does not depend on the choice of the positively oriented chart.

Let $\ell: U \to \mathbb{R}^d$ be another positively oriented chart (we may assume it has the same domain).

↻

$$\mu_x = H_d(j)\circ H_d(h^{-1})(v)$$

$$\Big(= H_d(j)\circ H_d(h^{-1}\circ \ell)\circ H_d(\ell^{-1})(v)\Big)$$

$$= H_d(j)\circ H_d(\ell^{-1})\circ H_d(\ell\circ h^{-1})(v)$$

To show: $H_d(\ell\circ h^{-1}) = \mathrm{id} : H_d(\mathbb{R}^d, \mathbb{R}^d\setminus\{0\}) \longrightarrow H_d(\mathbb{R}^d, \mathbb{R}^d\setminus\{0\})$.

$\ell\circ h^{-1} : \mathbb{R}^d \xrightarrow{\cong} \mathbb{R}^d$ induces a homeomorphism which is smooth except (possibly) at ∞.

$$\begin{tikzcd} g\colon \mathbb{R}^d_+ \arrow[r, "\cong"] \arrow[d, phantom, sloped, "\cong"] & \mathbb{R}^d_+ \arrow[d, phantom, sloped, "\cong"] \\ S^d & S^d \end{tikzcd}$$

By the lemma in section 2.3, AT I, p.40, the degree of g is $\operatorname{sign}\det D_0 g \in \{\pm 1\}$.

Note that the inclusion $\mathbb{R}^d \hookrightarrow \mathbb{R}^d_+ \cong S^d$ is the inverse of a chart. Computing $\det D_0 g$ in these local coordinates yields

$$\operatorname{sign}\det D_0 g = \operatorname{sign} J_0\, \ell\circ h^{-1} = 1.$$

The proof of the lemma I refer to only uses smoothness at one point. So the whole argument here goes through without further changes. But my claim in the lecture that $\ell\circ h^{-1}$ extends to a diffeo on the one-point compactification was incorrect. Sorry for my mistake!

Consider the diagram

$$\begin{tikzcd} H_d(\mathbb{R}^d, \mathbb{R}^d\setminus\{0\}) \arrow[rr, "H_d(\ell\circ h^{-1})"] \arrow[d, "H_d(K)"', "\cong"] & & H_d(\mathbb{R}^d, \mathbb{R}^d\setminus\{0\}) \arrow[d, "H_d(K)", "\cong"'] \\ H_d(K(\mathbb{R}^d), K(\mathbb{R}^d\setminus\{0\})) \arrow[d] & \circlearrowleft & H_d(K(\mathbb{R}^d), K(\mathbb{R}^d\setminus\{0\})) \arrow[d] \\ H_d(\mathbb{R}^d_+, \mathbb{R}^d_+\setminus\{0\}) & & H_d(\mathbb{R}^d_+, \mathbb{R}^d_+\setminus\{0\}) \\ H_d(\mathbb{R}^d_+) \arrow[u] \arrow[rr, "H_d(g) = \mathrm{id}"] & & H_d(\mathbb{R}^d_+) \arrow[u] \end{tikzcd}$$

Thus $H_d(\ell\circ h^{-1}) = \mathrm{id}$.

The family (μ_x) is a homological orientation.

(Check local triviality!) This shows "$\Rightarrow$".

Assume now that a homological orientation $(\mu_x)_{x\in M}$ is given.

We declare a chart $h\colon \underset{\substack{\rotatebox{90}{$\in$}\\ x}}{U} \longrightarrow \mathbb{R}^d$, $h(x)=0$, to be positively oriented if

$$\underset{\substack{\rotatebox{90}{$\in$}\\ \upsilon}}{H_d(\mathbb{R}^d, \mathbb{R}^d\setminus\{0\})} \xrightarrow[\cong]{H_d(h^{-1})} H_d(U, U\setminus\{x\}) \xrightarrow[\cong]{\text{exc.}} H_d(M, M\setminus\{x\})$$

maps υ to μ_x. By a similar argument as before, the set of positively oriented charts defines an orienting atlas. ☒

From the fact that $\mathbb{F}_2 \cong H_d(M, M\setminus\{x\}; \mathbb{F}_2)$ has a unique generator follows:

Prop.: Every manifold is $\mathbb{F}_2$-orientable.

Lemma: Let M be a d-dimensional manifold.

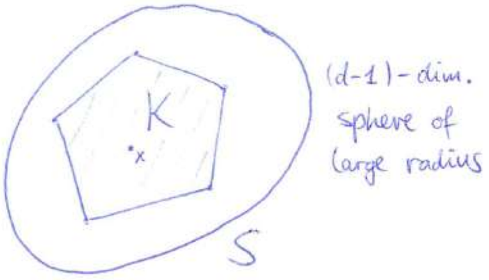
Let $K\subset M$ be compact. Then:

1) $H_i(M, M\setminus K; R) = 0$ for $i>d$.

2) If $u\in H_d(M, M\setminus K; R)$ is sent to $0\in H_d(M, M\setminus\{x\}; R)$ for every $x\in K$, then $u=0$.

3) If (μ_x) is an R-orientation, then there is a unique $\mu_K \in H_d(M, M\setminus K; R)$ mapping to μ_x for every $x\in K$.

Proof: 1. step: $M=\mathbb{R}^d$, K convex.



$$\left.\begin{array}{l} S\subset M\setminus K\\ S\subset M\setminus\{x\}\end{array}\right\}\text{ are deformation retracts}$$

(we are using the convexity here)

$$\Longrightarrow \quad \begin{tikzcd} H_i(S;R) \arrow[r, "\cong"] \arrow[dr, "\cong"', "\circlearrowright"] & H_i(M\setminus K;R) \arrow[d] \\ & H_i(M\setminus\{x\};R) \end{tikzcd}$$

↺

Considering the LES for $(M, M\setminus K)$ and $(M, M\setminus\{x\})$ and applying the 5-lemma, we obtain isomorphisms

$$H_i(M, M\setminus K; R) \xrightarrow{\cong} H_i(M, M\setminus\{x\}; R) \cong \begin{cases} R, & i=d, \\ 0, & i\neq d, \end{cases}$$

from which 1) and 2) and 3) follow.

2. Step: If the claim is true for K_1, K_2 and $K_1 \cap K_2$, then also for $K_1 \cup K_2$.

For 1) consider the MV-sequence of $(M; M\setminus K_1, M\setminus K_2)$:

$$H_{i+1}(M, M\setminus(K_1\cap K_2); R) \longrightarrow H_i(M, M\setminus(K_1\cup K_2); R) \longrightarrow \begin{matrix} 0 \\ \| \\ H_i(M, M\setminus K_1; R) \\ \oplus \\ H_i(M, M\setminus K_2; R) \\ \| \\ 0 \end{matrix} \longrightarrow H_i(M, M\setminus(K_1\cap K_2); R) \longrightarrow \cdots$$

For $i=d$,

$$\begin{tikzcd} H_d(M, M\setminus(K_1\cup K_2); R) \arrow[r, hook] & \begin{matrix} H_d(M, M\setminus K_1; R) \\ \oplus \\ H_d(M, M\setminus K_2; R) \end{matrix} \\ u \arrow[r, mapsto] & (u_1, u_2) \end{tikzcd}$$

is injective.

If u is mapped to $0 \in H_d(M, M\setminus\{x\}; R)$ $\forall x \in K_1 \cup K_2$, then u_i is mapped to $0 \in H_d(M, M\setminus\{x\}; R)$ $\forall x \in K_i$, so by assumption on K_i, we get $u_i = 0$.

By injectivity, $u = 0$. $\Longrightarrow$ 2). Ad 3): μ_{K_i} maps to $\mu_{K_1\cap K_2}$ under $H_d(M, M\setminus K_i; R) \to H_d(M, M\setminus(K_1\cap K_2); R)$ by uniqueness. Hence $(\mu_{K_1}, \mu_{K_2}) \mapsto 0$ in the MV-sequence. Thus (μ_{K_1}, μ_{K_2}) comes from an element $\mu_{K_1\cup K_2} \in H_d(M, M\setminus(K_1\cup K_2); R)$.

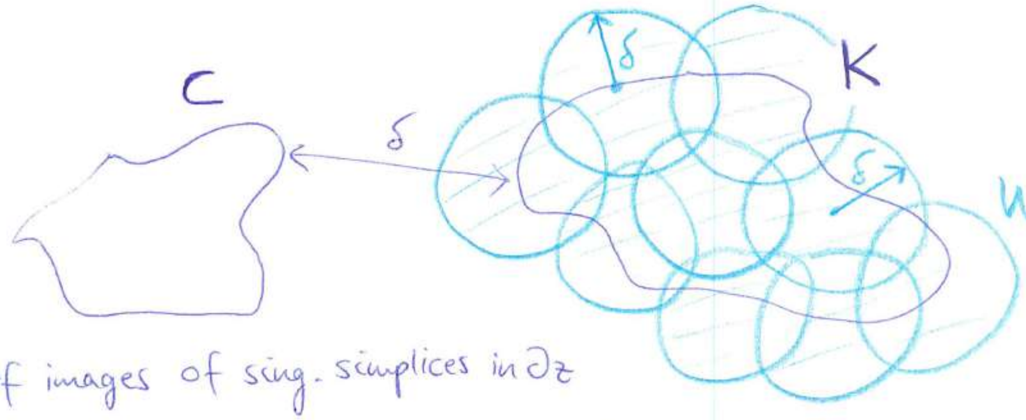
3. step: The claim is true for $M = \mathbb{R}^d$ and K being a finite union of convex sets.

This follows from step 1 and 2 and the fact that an intersection of convex sets is convex.

25.1.

4. step: The claim is true for $M = \mathbb{R}^d$ and arbitrary compact $K \subseteq \mathbb{R}^d$.

Let $\underset{\substack{\| \\ [z]}}{\alpha} \in H_i(\mathbb{R}^d, \mathbb{R}^d \setminus K; R)$. Then ∂z is a singular $(d-1)$-chain in $\mathbb{R}^d \setminus K$.



$C :=$ Union of images of sing. simplices in ∂z

Since C and K are compact, there is $\delta > 0$ with
$d(x,y) > \delta$ for all $x \in C$, $y \in K$.

By compactness of K we can cover K by finitely many balls with radius δ and center in K.

Obviously, we have. ($W :=$ Union of the balls ◍)

$$\alpha \in \operatorname{im}\left(H_i(\mathbb{R}^d, \mathbb{R}^d \setminus W; R) \longrightarrow H_i(\mathbb{R}^d, \mathbb{R}^d \setminus K; R) \right)$$

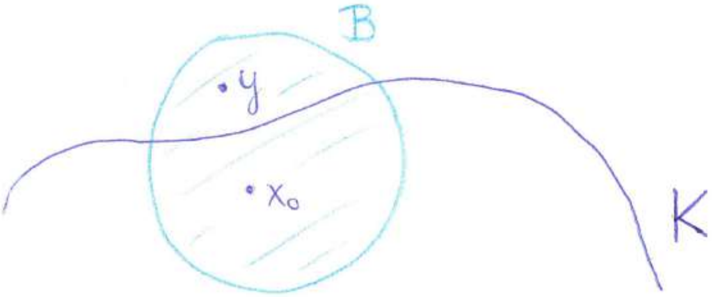
(since ∂z is a chain in $\mathbb{R}^d \setminus W$).

By step 3, $H_i(\mathbb{R}^d, \mathbb{R}^d \setminus W; R) = 0$ for $i > d$.

Hence the first statement follows.

↺

In view of step 3, it suffices for the 2nd statement to show that:
If the image $\alpha_x \in H_d(M, M\setminus\{x\}; R)$ of α is zero for all $x \in K$, then already $\alpha_x = 0$ for all $x \in W$ holds.



$$\begin{tikzcd} H_d(\mathbb{R}^d, \mathbb{R}^d\setminus W; R) \arrow[r] & H_d(\mathbb{R}^d, \mathbb{R}^d\setminus B; R) \arrow[r, "\cong"] & H_d(\mathbb{R}^d, \mathbb{R}^d\setminus\{x_0\}; R) \\ \alpha \arrow[r, mapsto] & \alpha_B \arrow[r, mapsto] & \alpha_{x_0} = 0 \end{tikzcd}$$

If $\alpha_{x_0} = 0$, $x_0 \in K$, then $\alpha_B = 0$.
Since α_y, $y \in B$, is the image of α_B, also $\alpha_y = 0$ $\forall y \in B$.
Doing this for every ball in W, we obtain $\alpha_x = 0$ $\forall x \in W$. $\Rightarrow$ 2nd statement.

Ad existence statement:
Choose a large enough ball E containing K.
The map $H_d(\mathbb{R}^d, \mathbb{R}^d\setminus E; R) \xrightarrow{\cong} H_d(\mathbb{R}^d, \mathbb{R}^d\setminus\{x\}; R)$ is an isomorphism for $x \in E$. For each generator $\nu \in H_d(\mathbb{R}^d, \mathbb{R}^d\setminus E; R)$ define

$$A_\nu := \{ x \in E \mid \nu \text{ maps to } \mu_x \}.$$

Then $\bigcup_{\nu \text{ generator}} A_\nu = E$.

↗

Each subset A_ν is open by local triviality of (μ_x):

Let $x_0 \in A_\nu \subset E$. Then there is a neighborhood U around x_0 so that

$$\begin{tikzcd} H_d(\mathbb{R}^d, \mathbb{R}^d \setminus U; R) \arrow[r, "\cong"] & H_d(\mathbb{R}^d, \mathbb{R}^d \setminus \{x\}; R) \\ \mu_U \arrow[r, mapsto] & \mu_x \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

μ_x, $x \in U$, come from an element μ_U. W.l.o.g. we may assume that U is a ball around x_0.

Since ν maps to μ_{x_0}, thus to μ_U, it maps to μ_x for all $x \in U$.

By connectedness of E, there is a unique generator $\nu \in H_d(\mathbb{R}^d, \mathbb{R}^d \setminus E; R)$ that maps to μ_x $\forall x \in E$.

The image of ν in $H_d(\mathbb{R}^d, \mathbb{R}^d \setminus K; R)$ is the desired element μ_K.

Step 5: The statements are true if K is contained in an open subset of M which is homeomorphic to $\mathbb{R}^d$.

Note that $H_i(M, M\setminus K; R) \xleftarrow[\text{excision}]{\cong} H_i(U, U\setminus K; R)$, then use **Step** 4.

Step 6: By choosing charts around each $x \in K$, we can write K as a finite union of compact subsets each of which – as well as their intersections – satisfy the requirements of step 5. With step 2 we finish the proof. ⊠

Notation: From now on, we denote the relative homology group $H_i(M, M \setminus K; R)$ by $H_i(M|K; R)$ and if $K = \{x\}$ by $H_i(M|x; R)$.

If $\mu \in H_i(M|L; R)$ and $K \subset L$, then we denote the image of μ under

$$H_i\left((M, M\setminus L) \xrightarrow{\text{inclusion}} (M, M \setminus K) \right)$$

by $\mu|_K$. Sometimes we refer to the induced map as restriction.

Remark: Implicit in the previous proof is that around every $x \in M$ there is an open neighborhood U s.th.

$$\begin{tikzcd} H_d(M|U; R) \arrow[r] & H_d(M|x; R) \\ \mu \arrow[r, mapsto] & \mu|_{\{x\}} \end{tikzcd}$$

is an isomorphism. Just take a chart $h: V \xrightarrow{\cong} \mathbb{R}^d$ around x with $h(x) = 0$ and $U = h^{-1}(\underset{\substack{\uparrow \\ \left(\text{unit ball around } 0\right)}}{B})$.

The isomorphism then follows from:

$$\begin{tikzcd} H_d(M|U; R) \arrow[r] & H_d(M|x; R) \\ H_d(V|U; R) \arrow[u, "\text{exc.}", "\cong"'] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & H_d(V|x; R) \arrow[u, "\cong"] \arrow[d, "\cong"'] \\ H_d(\mathbb{R}^d|B; R) \arrow[r, "\cong"] & H_d(\mathbb{R}^d|0; R) \end{tikzcd}$$

↗

Thm.: Let M be a closed, connected, d-dim. mfd.

Then: 1) If M is R-orientable,

$$H_d(M;R) \longrightarrow H_d(M|x;R)$$

is an isomorphism for every $x \in M$.

2) If M is not R-orientable, $\mu \longmapsto \mu_{|\{x\}}$ is still injective with image isomorphic to $\{ r \in R \mid 2r = 0 \}$.

3) $H_i(M;R) = 0 \quad \forall i > d$.

Proof: Choose $K = M$ in the previous lemma:

The surjectivity in 1) follows from the 3rd statement of the Lemma.

Ad injectivity: By the previous remark, each $x \in M$ has a neighborhood U_x such that

$$\ker\left(H_d(M;R) \longrightarrow H_d(M|y;R) \right)$$
$$||$$
$$\ker\left(H_d(M;R) \longrightarrow H_d(M|U_x;R) \right)$$

for all $y \in U_x$. So by connectedness of M, if

$\mu \longmapsto \mu_{|\{x_0\}} = 0$ is mapped to zero for one point $x_0 \in M$,

then $\mu \longmapsto \mu_{|\{x\}} = 0$ is mapped to zero for every $x \in M$.

By the second statement of the previous Lemma, $\mu = 0$.

$\rightsquigarrow$ Injectivity in 1) and 2).

↺

For the image statement in 2) we refer to Hatcher, p.236.

3) follows from the first statement of the Lemma.

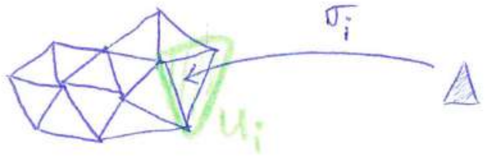
Def.: In the above setting, a generator of $H_d(M;R)$ is called a fundamental class, denoted by $[M]$.

Rem.: An R-orientation determines a fundamental class and vice versa.

How to visualize the fundamental class?

Assume that M is triangulated. Then:

$$H_d(M) \ni [M] = \sum_{j=1}^{m} k_j \sigma_j \, , \quad \sigma_j : \Delta_d \xrightarrow{\cong} M$$ affine isom. onto a d-simplex of the triangulation



Think of σ_i as a "local orientation" of the image.

26.1.

$$\begin{tikzcd}
H_d(M) \arrow[r, "\cong"] & H_d(M|x) & H_d(U_i|x) \arrow[l, "\cong"'] \\
{[M] = [\sum_j k_j \sigma_j]} \arrow[u, phantom, "\in"] \arrow[r, mapsto] & k_i[\sigma_i] \arrow[u, phantom, "\in"] & \sigma_i \arrow[u, phantom, "\in"] \\
 & \text{generator} & \text{generator}
\end{tikzcd} \quad , \; x \in U_i$$

$\Rightarrow \quad k_i \in \{\pm 1\}$

So the fundamental class of a triangulated closed manifold of Dimension d is a "signed sum" of all d-simplices in the triangulation.

In general, if $[\sum_{i=1}^{m} a_i \sigma_i]$ represents the fundamental class of M, then

$$\bigcup_{i=1}^{m} \operatorname{im}(\sigma_i) = M .$$

Otherwise $H_d(M) \longrightarrow H_d(M|x)$ would be the zero map for $x \notin \bigcup_{i=1}^{m} \operatorname{im}(\sigma_i)$.

10.2 The degree of a map between closed manifolds

Def.: Let M, N be closed, connected, oriented, d-dimensional manifolds.
Let $f: M \to N$ be a map.
The degree of f is the integer $\deg(f)$ with

$$H_d(f)([M]) = \deg(f) \cdot [N] .$$

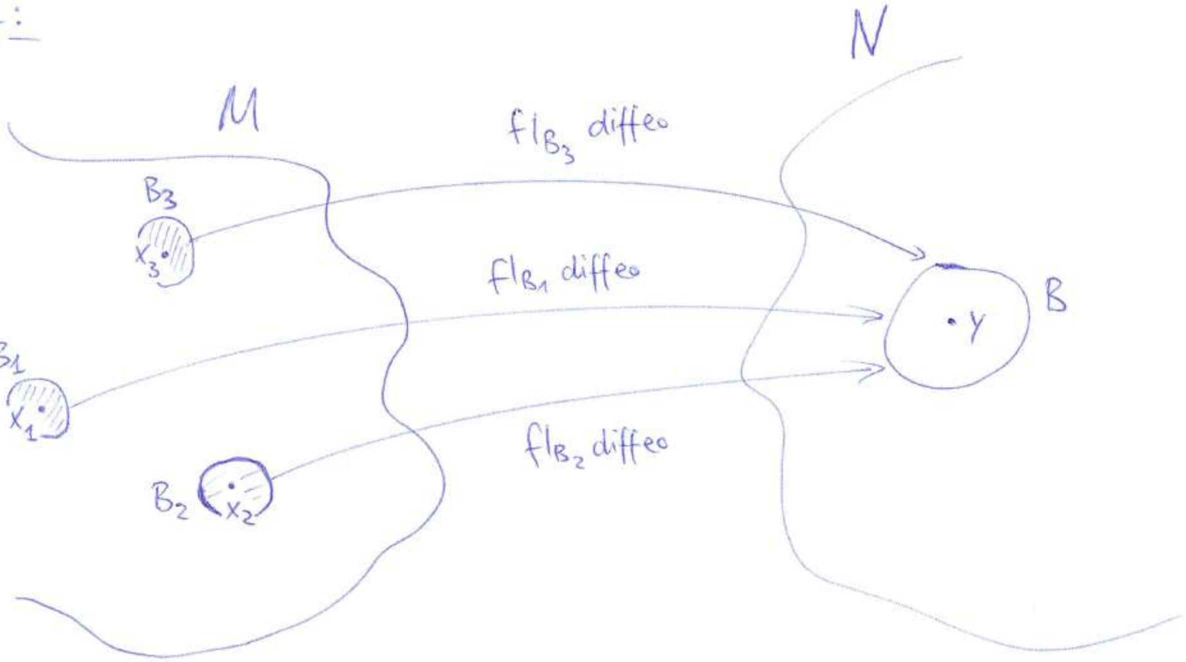
Rem.:
- If $\deg(f) \neq 0$, then f is surjective.
- $\deg(f \circ g) = \deg(f) \cdot \deg(g)$
- $\deg(\overline{M} \xrightarrow{f} N) = -\deg(f: M \to N)$, where $\overline{M}$ carries the opposite orientation.

Rem.: Let V, W be oriented d-dim. vector spaces.
Let $f: V \to W$ be a linear isomorphism.
Then $\operatorname{sign} \det(f) := \operatorname{sign} \det M_{B_V B_W}(f) \in \{\pm 1\}$
for positively oriented bases B_V, B_W of V and W, respectively, does not depend on the choice of B_V, B_W. ↺

Theorem: Let M, N be closed, smooth, connected, oriented d-dim. manifolds. Let $f: M \to N$ be smooth. Let $y \in N$ be a regular value (i.e. $T_x f$ is an isomorphism for all $x \in f^{-1}(y)$). Then

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(\det T_x f) .$$

Proof:



$$\begin{tikzcd}
H_d(B_i | x_i) \arrow[r, "H_d(f)"] \arrow[d, "\text{chart } k_i"'] \arrow[dr, phantom, "\circlearrowleft"] & H_d(B|y) \arrow[d, "\text{chart } \ell"] \\
H_d(\mathbb{R}^d / 0) \arrow[r, "\cong", "H_d(\ell \circ f \circ k_i^{-1})"'] & H_d(\mathbb{R}^d | 0)
\end{tikzcd}$$

$$H_d(\ell \circ f \circ k_i^{-1}) = \left(\operatorname{sign} \det J_0(\ell \circ f \circ k_i^{-1}) \right) \cdot \mathrm{id}$$

[cf. proof of equivalence of homological and "tangential" orientation]

↗

Define μ_{B_i} and μ_B via

$$\begin{tikzcd}
\mu_{x_i} \arrow[r, phantom, "\in"] & H_d(M|x_i) & & H_d(N|y) \arrow[r, phantom, "\ni"] & \mu_y \\
& H_d(B_i|x_i) \arrow[u, "\cong"] \arrow[rr, "H_d(f|_{B_i})"'] \arrow[dl, phantom, sloped, "\ni"] & & H_d(B|y) \arrow[u, "\cong"'] \arrow[dr, phantom, sloped, "\ni"] & \\
\mu_{B_i} \arrow[uu, mapsto] & & & & \mu_B \arrow[uu, mapsto]
\end{tikzcd}$$

We have $H_d(f|_{B_i})(\mu_{B_i}) = (\text{sign} \det T_{x_i} f) \cdot \mu_B$.

$$\begin{tikzcd}
\bigoplus H_d(B_i|x_i) \arrow[rrr, "\bigoplus H_d(f|_{B_i})"] \arrow[dd, "\cong"] \arrow[dr, phantom, sloped, "\ni"] & & & H_d(B|y) \arrow[r, phantom, "\ni"] \arrow[dd, "\cong"'] & \mu_B \arrow[dd, mapsto] \\
& (\mu_{B_i}) \arrow[r, mapsto] \arrow[d, mapsto] & \sum\limits_{x_i \in f^{-1}(y)} (\text{sign} \det T_{x_i} f) \mu_B & & \\
\bigoplus H_d(M|x_i) \arrow[r, phantom, "\ni"] & (\mu_{x_i}) & \circlearrowright & H_d(N|y_\cdot) \arrow[r, phantom, "\ni"] & \mu_y \\
& {[M]} \arrow[u, mapsto] & & & {[N]} \arrow[u, mapsto] \\
H_d(M) \arrow[uu] \arrow[ur, phantom, sloped, "\ni"] \arrow[rrr, "H_d(f)"] & & & H_d(N) \arrow[uu, "\cong"] \arrow[ur, phantom, sloped, "\ni"] &
\end{tikzcd}$$

The above diagram concludes the proof. ▨

10.3 Poincare duality

Our goal is to show the following theorem:

Theorem (Poincare duality):

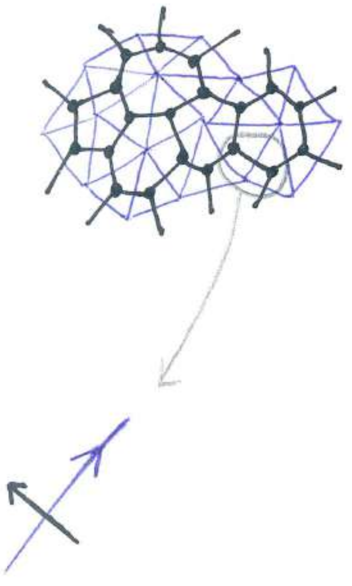
Let M be a closed connected R-oriented d-dim. manifold. The cap product with the fundamental class

$$P: \quad H^k(M;R) \longrightarrow H_{d-k}(M;R)$$

is an isomorphism for all k.

What is the intuition behind Poincare duality?

Assume M is triangulated.



Triangulation ⇝ dual CW-decomposition

$$\left\{\begin{matrix}\text{i-simplices of}\\ \text{the triangulation}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{(d-i)-cells of the}\\ \text{dual CW-decomposition}\end{matrix}\right\}$$

(d-i)-cell ↖ as a cochain:
on i-simplices
of the triangulation

$$\longmapsto \begin{cases} 1 & \text{matches orientation of } M \\ 0 & \text{no crossing} \\ -1 & \text{does not match orientation} \end{cases}$$

Exam dates: 8.3., 17.3., 18.3., 23.3.

A reminder on colimits (cf. exercise 1 on problem sheet 2):

We consider a directed set, i.e. a set I with a partial order "$\leq$" so that for any $x, y \in I$ there is $z \in I$ with $x \leq z$, $y \leq z$.

$(I, \leq)$ can be regarded as a category $\mathcal{C}_{(I,\leq)}$ with $\operatorname{ob}\mathcal{C}_{(I,\leq)} = I$ and

$$\operatorname{mor}_{\mathcal{C}_{(I,\leq)}}(x,y) = \begin{cases} *, & \text{if } x \leq y, \\ \emptyset, & \text{else} \end{cases}.$$

In the sequel we'll consider colimits of functors $\mathcal{F}: \mathcal{C}_{(I,\leq)} \to R\text{-Mod}$.

We write $\operatorname{colim}_{i \in I} \mathcal{F}(i)$ instead of $\operatorname{colim}_{\mathcal{C}_{(I,\leq)}} \mathcal{F}$.

If $(I, \leq) = (\mathbb{N}, \leq)$ we also write $\operatorname{colim}_{i \to \infty} \mathcal{F}(i)$.

- colim over directed sets is exact, i.e. if $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ are natural transformations of functors $\mathcal{C}_{(I,\leq)} \to R\text{-mod}$ s.th.

$$0 \longrightarrow \mathcal{F}_1(i) \longrightarrow \mathcal{F}_2(i) \longrightarrow \mathcal{F}_3(i) \longrightarrow 0$$

is exact, then

$$0 \longrightarrow \operatorname{colim}_{i\in I} \mathcal{F}_1(i) \longrightarrow \operatorname{colim}_{i\in I} \mathcal{F}_2(i) \longrightarrow \operatorname{colim}_{i\in I} \mathcal{F}_3(i) \longrightarrow 0$$

is exact.

- If all $\mathcal{F}(i)$'s are submodules of an R-module N and the maps $\mathcal{F}(i \leq j): \mathcal{F}(i) \to \mathcal{F}(j)$ are inclusions, then

$$\operatorname{colim}_{i \in I} \mathcal{F}(i) = \bigcup_{i \in I} \mathcal{F}(i).$$

This follows immediately from the universal property. ↺

- If $J \subseteq I$ and for every $x \in I$ there is $y \in J$ with $x \leq y$, then

$$\operatorname{colim}_{i \in I} F(i) \xleftarrow{\cong} \operatorname{colim}_{j \in J} F(j) .$$

Def.: Let $C^*_c(X;R)$ be the subchain complex of $C^*_{sing}(X;R)$ given by singular cochains

$$f : S_n(X) \longrightarrow R$$

for which there is a compact subset $K_f \subset X$ with

$$\left.\begin{array}{l} \sigma : \Delta_n \to X \\ \emptyset = im(\sigma) \cap K_f \end{array}\right\} \Rightarrow f(\sigma) = 0 .$$

The singular cohomology of X (with coefficients in R) with compact support is

$$H^*_c(X;R) := H^*(C^*_c(X;R)) .$$

(For compact X, we have $H^*_c(X;R) = H^*(X;R)$.)

Let $\mathcal{C}(X)$ be the set of compact subsets of X ordered by inclusion. The inclusions $C^*_{sing}(X, X \setminus K; R) \hookrightarrow C^*_c(X;R)$, $K \in \mathcal{C}(X)$, induce an isomorphism

$$\underbrace{\operatorname{colim}_{K \in \mathcal{C}(X)} C^*_{sing}(X, X \setminus K; R)}_{\overset{\shortparallel}{\bigcup_{K \in \mathcal{C}(X)} C^*_{sing}(X, X \setminus K; R)}} \xrightarrow{\cong} C^*_c(X;R) .$$

↗

By flatness, we obtain an isomorphism

$$\operatorname*{colim}_{K \in \mathcal{C}(X)} H^*(X \mid K; R) \xrightarrow{\cong} H^*_c(X; R) .$$

Let M be a d-dimensional oriented manifold with R-orientation (μ_x).

For each $K \in \mathcal{C}(M)$ there is exactly one homology class μ_K,

$\mu_K \in H_d(M \mid K; R)$ with $\mu_{K \mid \{x\}} = \mu_x \quad \forall x \in K$.

If $K \subset L$, then $\mu_{L \mid K} = \mu_K$. Thus

$$\begin{tikzcd}
H^{d-i}(M \mid K; R) \arrow[r, "\mu_K \cap \_"] \arrow[d] & H_i(M; R) \arrow[d, equal] \\
H^{d-i}(M \mid L; R) \arrow[r, "\mu_L \cap \_"] & H_i(M; R)
\end{tikzcd}$$

commutes.

Taking the colimit over $\mathcal{C}(X)$ we obtain a homomorphism

$$H^{d-i}_c(M; R) \cong \operatorname*{colim}_{K \in \mathcal{C}(X)} H^{d-i}(M \mid K; R) \longrightarrow H_i(M; R)$$

called the <u>Poincare duality homomorphism</u> – denoted by $P_i(M) = P_i$.

<u>Theorem</u> (Poincare duality – general):

For every d-dimensional, R-oriented manifold M,

$$P_i = P_i(M) : \quad H^{d-i}_c(M; R) \longrightarrow H_i(M; R)$$

is an isomorphism for all i.

↺

From now on, R is fixed and we write $H_c^*(-)$, $H_*(-)$ for $H_c^*(-;R)$, $H_*(-;R)$.

Let U, V be open with $M = U \cup V$. For every $K \in \mathcal{C}(U)$ and $L \in \mathcal{C}(V)$ we have a MV sequence

$$\cdots \longrightarrow H^k(M \mid K \cap L) \longrightarrow \begin{matrix} H^k(M \mid K) \\ \oplus \\ H^k(M \mid L) \end{matrix} \longrightarrow H^k(M \mid K \cup L) \longrightarrow H^{k+1}(M \mid K \cap L) \longrightarrow \cdots$$

Replacing $H^k(M \mid K \cap L)$ and $H^k(M \mid K) \oplus H^k(M \mid L)$ by $H^k(U \cap V \mid K \cap L)$ and $H^k(U \mid K) \oplus H^k(V \mid L)$ via excision we obtain a LES

$$\cdots \longrightarrow H^k(U \cap V \mid K \cap L) \longrightarrow \begin{matrix} H^k(U \mid K) \\ \oplus \\ H^k(V \mid L) \end{matrix} \longrightarrow H^k(M \mid K \cup L) \longrightarrow H^{k+1}(U \cap V \mid K \cap L) \longrightarrow \cdots$$

If (K_n) and (L_n) are increasing sequences in $\mathcal{C}(U)$ and $\mathcal{C}(V)$ with $\bigcup_{n \in \mathbb{N}} K_n = U$ and $\bigcup_{n \in \mathbb{N}} L_n = V$, then $(K_n \cup L_n)$ is an increasing sequence in $\mathcal{C}(M)$ with $\bigcup_{n \in \mathbb{N}} (K_n \cup L_n) = M$. Taking the colimit $n \to \infty$ of this LES we get a LES

$$\cdots \longrightarrow H_c^k(U \cap V) \longrightarrow H_c^k(U) \oplus H_c^k(V) \longrightarrow H_c^k(M) \longrightarrow H_c^{k+1}(U \cap V) \longrightarrow \cdots$$

$\nearrow$

Commutativity lemma : If $M = U \cup V$, then

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(U\cap V) \arrow[r] \arrow[d, "P(U\cap V)"'] & H^k_c(U) \oplus H^k_c(V) \arrow[r] \arrow[d, "P(U)\oplus -P(V)"'] & H^k_c(M) \arrow[r] \arrow[d, "P(M)"'] & H^{k+1}_c(U\cap V) \arrow[r] \arrow[d, "P(U\cap V)"'] & \cdots \\
\cdots \arrow[r] & H_{d-k}(U\cap V) \arrow[r] & H_{d-k}(U) \oplus H_{d-k}(V) \arrow[r] & H_{d-k}(M) \arrow[r] & H_{d-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

commutes up to sign.

(See Hatcher, p. 246 for a proof.)

Proof of Poincare duality : We use two inductive steps.

(A) If M is the union of open subsets U and V and $P(U)$, $P(V)$ are isomorphisms, then also $P(M)$ is an iso.

This follows from the diagram above and the 5-lemma.

(B) If M is an increasing union of open subsets $U_1 \subset U_2 \subset \ldots$, $\bigcup_{n\in\mathbb{N}} U_n = M$, and each $P(U_i)$ is an iso., then $P(M)$ is an iso.

By excision,

$$H^k_c(U_i) \cong \operatorname*{colim}_{K\in\mathcal{C}(U_i)} H^k(U_i|K) \xrightarrow{\cong} \operatorname*{colim}_{K\in\mathcal{C}(U_i)} H^k(M|K).$$

Further, $\bigcup_{i\in\mathbb{N}} \mathcal{C}(U_i) = \mathcal{C}(M)$. Thus, $\operatorname*{colim}_{i\in\mathbb{N}} H^k_c(U_i) \xrightarrow{\cong} H^k_c(M)$.

We have $\operatorname*{colim}_{i\to\infty} C^{sing}_*(U_i) \xrightarrow{\cong} C^{sing}_*(M)$ since the union of images of singular simplices of a chain in $C^{sing}_*(M)$ lies in one of the sets U_i.

↺

By flatness, $\operatorname{colim}_{i\to\infty} H_*(U_i) \xrightarrow{\cong} H_*(M)$.

As a colimit of the $P(U_i)$'s, $P(M)$ is an iso.

We conclude the proof in 3 steps.

1) $M = \mathbb{R}^d$:

$$
\begin{aligned}
H^i_c(\mathbb{R}^d) &\cong \operatorname*{colim}_{K\in \mathcal{C}(\mathbb{R}^d)} H^i(\mathbb{R}^d, \mathbb{R}^d \setminus K) \\
&\cong \operatorname*{colim}_{n\to\infty} H^i(\mathbb{R}^d, \mathbb{R}^d \setminus B_0(n)) \qquad \circledast \\
&\cong \quad H^i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \\
&\cong \begin{cases} R, & i=d, \\ 0, & i\neq d. \end{cases}
\end{aligned}
$$

$$H_{d-i}(\mathbb{R}^d) \cong H_{d-i}(*) \cong \begin{cases} R, & i=d, \\ 0, & i\neq d. \end{cases}$$

We only have to check that

$$P(\mathbb{R}^d) : H^d_c(\mathbb{R}^d) \longrightarrow H_0(\mathbb{R}^d)$$

is an isomorphism.

Since the structure maps in the colimit $\circledast$ are all iso's we only have to verify that

$$H^d(\mathbb{R}^d, \mathbb{R}^d \setminus B_0(1)) \xrightarrow{\mu_{B_0(1)} \cap -} H_0(\mathbb{R}^d)$$

is an iso.

By the universal coefficient theorem,

$$
\begin{aligned}
H^d(\mathbb{R}^d, \mathbb{R}^d \setminus B_0(1)) &\xrightarrow{\cong} R \\
v &\longmapsto \langle v, \mu_{B_0(1)} \rangle
\end{aligned}
$$

and

$$
\begin{aligned}
H_0(\mathbb{R}^d) &\xrightarrow{\cong} R \\
v &\longmapsto \langle 1_{H_0(\mathbb{R}^d)}, v \rangle
\end{aligned}
$$

are iso's. ↗

Thus, $v \longmapsto \mu_{B_0(1)} \cap v$ iso

$$\Updownarrow$$

$$v \longmapsto \langle 1, \mu_{B_0(1)} \cap v \rangle \quad \text{iso}$$
$$\shortparallel$$
$$\langle 1 \cup v, \mu_{B_0(1)} \rangle$$
$$\shortparallel$$
$$\langle v, \mu_{B_0(1)} \rangle$$

2) $P(M)$ is an iso for an arbitrary open subset $M \subseteq \mathbb{R}^d$. <u>2.2.</u>

M can be written as a countable union of bounded convex subsets $\mathcal{U}_1, \mathcal{U}_2, \ldots$ (e.g. balls).

Set $V_i = \bigcup_{j<i} \mathcal{U}_j$. Both V_i and $\mathcal{U}_i \cap V_i = \bigcup_{j<i} \mathcal{U}_i \cap \mathcal{U}_j$ are the union of $i-1$ bounded convex open subsets, so by induction on the number of such sets in a cover we may assume that $P(V_i)$ and $P(\mathcal{U}_i \cap V_i)$ are isomorphisms.

By 1) $P(\mathcal{U}_i)$ is an iso.

Hence $P(\mathcal{U}_i \cup V_i) = P(V_{i+1})$ is an isomorphism by (A).

Since M is the increasing union of the V_i, by (B) $P(M)$ is an iso.

3) Conclusion of the proof:

Since a manifold has a countable basis of its topology, there is a countable family $(\mathcal{U}_i)$ of open sets s.th. $M = \bigcup_{i \in \mathbb{N}} \mathcal{U}_i$, $\mathcal{U}_i \cong \mathbb{R}^d \ \forall i$.

By (2) and (A), $P(V_j)$ for $V_j = \bigcup_{i<j} \mathcal{U}_i$ is an iso.

By (B), $P(M)$ is an iso. ☒ ⤴

For a closed oriented manifold M we now study the consequences of Poincare duality on the cup pairing:

$$H^k(M;R) \times H^{d-k}(M;R) \xrightarrow{\cup} H^d(M;R) \xrightarrow{\langle \cdot , [M] \rangle} R$$

Such a bilinear pairing $A \times B \to R$ is said to be <u>non-singular</u>, if the maps $A \longrightarrow \mathrm{Hom}(B;R)$ and $B \longrightarrow \mathrm{Hom}(A;R)$ obtained by viewing the pairing as a function of each variable seperately are both isomorphisms.

<u>Prop.:</u> The cup product - pairing is non-singular for closed R-orientable manifolds when R is a field or when $R = \mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out.

<u>Proof:</u> Consider the composition

$$H^{n-k}(M;R) \xrightarrow{h} \mathrm{Hom}_R(H_{n-k}(M;R);R) \xrightarrow{D^*} \mathrm{Hom}_R(H^k(M;R),R),$$

where h is the map appearing in the universal coefficient theorem induced by evaluation of cochains on chains, and D^* is the Hom-dual of the Poincare duality map $D: H^k(M;R) \to H_{n-k}(M;R)$.
The composition $D^* \circ h$ sends $\psi \in H^{n-k}(M;R)$ to the homomorphism

$$\varphi \longmapsto \psi([M] \cap \varphi) = (\varphi \cup \psi)[M].$$

Via h we consider ψ as the element $h(\psi) \in \mathrm{Hom}(H_{n-k}(M;R),R)$.

$\leadsto \quad D^*: \psi \longmapsto (\varphi \mapsto \psi([M] \cap \varphi))$

↗

For field coefficients or for integer coefficients with torsion factored out, h is an isomorphism.
Non-singularity of the pairing in one of its variables is equivalent to D being an isomorphism. ☒

Corollary: If M is a closed connected orientable n-manifold, then for each $\alpha \in H^k(M;\mathbb{Z})$ of infinite order that is not a proper multiple of another element, there exists an element $\beta \in H^{n-k}(M;\mathbb{Z})$ such that $\alpha \cup \beta$ is a generator of $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$.
With coefficients in a field, the same is true for any $\alpha \neq 0$.

Proof: The hypothesis on α means that it generates a $\mathbb{Z}$-summand of $H^k(M;\mathbb{Z})$. In this case, there is a homomorphism

$$\varphi: H^k(M;\mathbb{Z}) \longrightarrow \mathbb{Z} \quad , \quad \varphi(\alpha) = 1 .$$

By the non-singularity of the cup product pairing, φ is realized by taking the cup-product with an element $\beta \in H^{n-k}(M;\mathbb{Z})$ and evaluating on $[M]$. So $\alpha \cup \beta$ generates $H^n(M;\mathbb{Z})$. ☒

Def.: Let M be a compact $4n$-dimensional R-oriented manifold. The symmetric, non-singular bilinear form

$$\begin{aligned} H^{2n}(M;R) \times H^{2n}(M;R) &\longrightarrow R \\ (u\ ,\ v) &\longmapsto \langle u \cup v, [M] \rangle \end{aligned}$$

is called <u>intersection form</u> (with coefficients in R).

↺

<u>Def.:</u> Let M be a closed connected oriented manifold of dimension $d = 4n$. Then the <u>signature</u> $\operatorname{sign}(M) \in \mathbb{Z}$ is defined as the signature of the intersection form of M.
If M is not connected, one defines its signature as the sum of the signatures of its path components.
if $4 \nmid \dim(M)$, we set $\operatorname{sign}(M) = 0$.

Recall that the signature of a symmetric bilinear form on a finitely generated $\mathbb{Z}$-module A is the signature, i.e. # positive eigenvalues − # negative eigenvalues, of the induced bilinear form on $A \otimes_{\mathbb{Z}} \mathbb{R}$.

<u>Example:</u> $M = \mathbb{C}P^{2n}$, $4n$-dim. manifold, $z \in H^2(\mathbb{C}P^{2n})$ generator.
Orient it by $[\mathbb{C}P^{2n}] \in H_{4n}(M)$ with $\langle z^{2n}, [\mathbb{C}P^{2n}] \rangle = 1$.
(This corresponds to the orientation coming from the complex structure.)
Then:

$$\begin{aligned} \overset{\mathbb{Z} \atop \cong}{H^{2n}(M)} \times H^{2n}(M) &\longrightarrow \mathbb{Z} \\ \langle z^{n}, z^{n} \rangle &\longmapsto \langle z^n \cup z^n, z^{2n} \rangle \end{aligned}$$

has signature 1. So $\operatorname{sign}(\mathbb{C}P^{2n}) = 1$.

↗

10.4 Some applications and computations

Example: The cup product structure of $H^*(\mathbb{C}P^n;\mathbb{Z})$ as a truncated polynomial ring $\mathbb{Z}[\alpha]/_{\alpha^{n+1}}$, $|\alpha|=2$ can be deduced from Poincare duality.

$\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induces an isomorphism on H^i for $i \leq 2n-2$, by induction on n, $H^{2i}(\mathbb{C}P^n;\mathbb{Z})$ is generated by α^i for $i<n$.

By the corollary above there is an $m \in \mathbb{Z}$ s.th.

$\alpha \cup m\alpha^{n-1} = m\alpha^n$ generates $H^{2n}(\mathbb{C}P^n;\mathbb{Z})$.

$\rightsquigarrow$ $m = \pm 1$ and $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/_{\alpha^{n+1}}$.

The same argument shows $H^*(\mathbb{H}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/_{\alpha^{n+1}}$, $|\alpha|=4$.

For $\mathbb{R}P^n$ we use $\mathbb{Z}_2$-coefficients to show that

$$H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/_{\alpha^{n+1}} \quad , |\alpha|=1 .$$

Theorem: Let M be a closed manifold, odd-dimensional.

Then $\chi(M)=0$.

Proof: M is $\mathbb{F}_2$-orientable. By Poincare duality and universal coefficients,

$$\dim_{\mathbb{F}_2} H_{d-i}(M;\mathbb{F}_2) = \dim_{\mathbb{F}_2} H^i(M;\mathbb{F}_2) = \dim_{\mathbb{F}_2} H_i(M;\mathbb{F}_2) .$$

$$\Rightarrow \chi(M) = \sum_{i=0}^{d} (-1)^i \dim_{\mathbb{F}_2} H_i(M;\mathbb{F}_2) = 0 .$$

⊠

↺

Prop.: A 3-dimensional closed connected manifold M with $H_1(M)=0$ is a homology sphere, i.e. $H_*(M) = H_*(S^3)$.

Proof: Universal coefficient theorem: $H^1(M) \cong \mathrm{Hom}(H_1(M); \mathbb{Z}) = 0$

Poincare duality: $H_2(M) \cong H^1(M) = 0$, $H_3(M) \cong \mathbb{Z}$. $\boxtimes$

Poincare conjecture:

A simply-connected homotopy sphere (integral homology sphere) is homeomorphic to a sphere.

Theorem (Poincare):

There is a compact, 3-dim., connected, orientable homology sphere, which is not homotopy equivalent to S^3.

Sketch of the proof: 8.2.

One can realize A_5 as a subgroup Γ of $SO(3)$ – the group of rotational symmetries of an icosahedron.

Algebra $\Rightarrow$ A_5 is perfect, i.e. $[A_5, A_5] = A_5$.

Consider
$$q : \underset{\substack{\| \\ \{\text{quaternions} \\ \text{of norm } 1\}}}{S^3} \longrightarrow SO(3)$$

given by
$$g \longmapsto \left(\begin{matrix} \underset{\substack{\| \\ \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k}}{\mathbb{R}^3} \longrightarrow \mathbb{R}^3 \\ x \longmapsto \underset{\substack{\uparrow \\ \text{multiplication} \\ \text{in } \mathbb{H}}}{gxg^{-1}} \end{matrix} \right).$$

↗

q is a surjective homomorphism with kernel $\ker(q) = \{+1,-1\}$.

Define $\tilde{\Gamma} = q^{-1}(\Gamma)$. We have $|\tilde{\Gamma}| = 2 \cdot |A_5| = 120$.

$\tilde{\Gamma}$ is also perfect:

$$\left\{ \begin{array}{l} [i,j] = i j i^{-1} j^{-1} = k \cdot k = -1 \\ q(i), q(j) \in \Gamma \end{array} \right.$$

$$\Rightarrow \quad \ker(q) \subset [\tilde{\Gamma}, \tilde{\Gamma}]$$

From $$1 \longrightarrow \ker(q) \longrightarrow [\tilde{\Gamma}, \tilde{\Gamma}] \xrightarrow{q} \overset{\Gamma}{\overset{\shortparallel}{[\Gamma, \Gamma]}} \longrightarrow 1$$

we get $|[\tilde{\Gamma}, \tilde{\Gamma}]| = |\ker(q)| \cdot |\Gamma| = 120$.

Thus $[\tilde{\Gamma}, \tilde{\Gamma}] = \tilde{\Gamma}$.

Define $\Sigma = S^3/\tilde{\Gamma}$. Σ is a closed orientable 3-manifold with universal covering $S^3 \xrightarrow{\text{proj}} \Sigma$. Hence $\pi_1(\Sigma) = \tilde{\Gamma}$.

By Hurewicz, $H_1(\Sigma) \cong \pi_1 / [\pi_1, \pi_1] = 0$.

By Poincare duality, Σ is a homology 3-sphere. ▨

10.5 Homological orientation and Poincare duality for manifolds with boundary

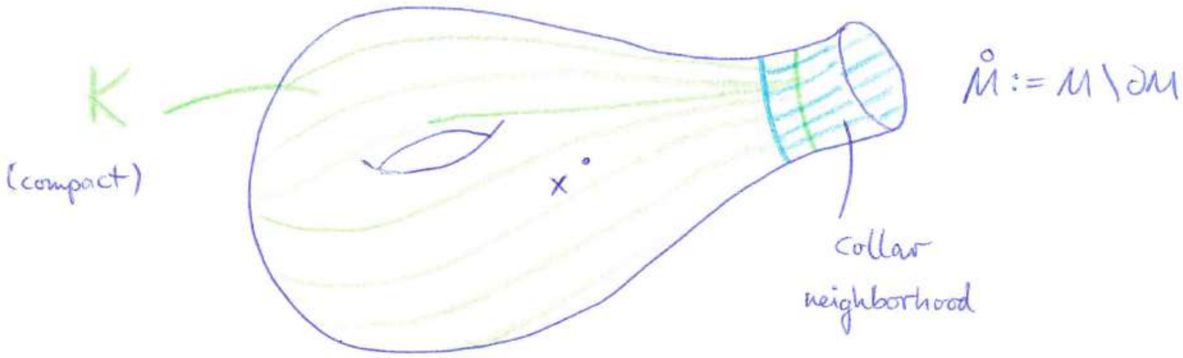
In the sequel let M denote a compact d-dimensional manifold with boundary ∂M. An R-orientation of M is an R-orientation $(\mu_x)_{x \in M \setminus \partial M}$ of $M \setminus \partial M$.

↺

For every R-orientation $(\mu_x)_{x\in M\setminus\partial M}$ there is a unique homology class $[M,\partial M]\in H_d(M,\partial M;R)$, called <u>fundamental class</u>, such that the restriction

$$\begin{tikzcd}
 & H_d(\mathring{M},\mathring{M}\setminus\{x\};R)\ni\mu_x \arrow[d, "\cong"'] \\
H_d(M,\partial M;R) \arrow[r] & H_d(M,M\setminus\{x\};R)
\end{tikzcd}\quad , \quad x\in M\setminus\partial M,$$

maps $[M,\partial M]$ to μ_x:



$$\begin{tikzcd}
 & & \mu_K \arrow[ddll, mapsto, bend right] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \\
 & & H_d(\mathring{M},\mathring{M}\setminus K;R) \arrow[ddll] \arrow[dd, "\cong"] & \\
\mu_x \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & & \\
H_d(\mathring{M},\mathring{M}\setminus\{x\};R) \arrow[d, "\cong"] & & H_d(M,M\setminus K;R) & \\
H_d(M|x;R) & & H_d(M,\partial M;R) \arrow[u, "\cong"'] & {[M,\partial M]} \arrow[uuuul, mapsto, bend right]
\end{tikzcd}$$

The image of $[M,\partial M]$ under

$$H_d(M,\partial M;R)\longrightarrow H_{d-1}(\partial M;R)$$

is a fundamental class of ∂M.

↗

Theorem (Poincare duality for manifolds with boundary):

Let M be an R-oriented compact d-dimensional manifold with boundary ∂M. Then the so-called Poincare duality homomorphism

$$\_\cap [M,\partial M] : H^{d-i}(M;R) \longrightarrow H_i(M,\partial M;R)$$
$$\_\cap [M,\partial M] : H^{d-i}(M,\partial M;R) \longrightarrow H_i(M;R)$$

is an isomorphism $\forall i$.

(Without proof.)

Theorem (Bordism invariance of signature):

Let M be a compact $(4d+1)$-dim. oriented manifold with boundary. Then $\mathrm{sign}(\partial M) = 0$.

Lemma: Let $s: V \times V \longrightarrow \mathbb{R}$ symmetric, non-singular bilinear form on a finite-dim. $\mathbb{R}$-vectorspace V.

Then $\mathrm{sign}(s) = 0$ if and only if there is a subspace $L \subset V$ with

$$\begin{cases} 2 \cdot \dim L = \dim V, \\ s(a,b) = 0 \quad \forall a,b \in L. \end{cases}$$

Proof: Assume first that $\mathrm{sign}(s) = 0$. Then there is a orthogonal basis $\{b_1, \ldots, b_n, c_1, \ldots, c_n\}$ with $s(b_i, b_i) = 1$ and $s(c_i, c_i) = 1$ $\forall i \in \{1, \ldots, n\}$. Then set $L = \mathrm{span}\{b_i + c_i \mid i = 1, \ldots, n\}$.

↺

Assume next that L as above exists.

Let $V_+ \subseteq V$ and $V_- \subseteq V$ be the maximal subspace s.th. $s|_{V_+\times V_+}$ is positive definite and $s|_{V_-\times V_-}$ is negative definite.

We have $\begin{cases} V_\pm \cap L = \{0\} \\ V_+ \oplus V_- = V \end{cases}$ by non-singularity.

$$\Rightarrow \quad \dim(V_\pm) + \dim(L) - \dim(V_\pm \cap L) \leq \dim(V)$$
$$\Rightarrow \quad \dim(V_\pm) \leq \dim(V) - \dim(L) = \tfrac{1}{2}\dim(V)$$
$$\Rightarrow \quad \dim(V_\pm) = \frac{1}{2}\dim(V)$$
$$\Rightarrow \quad \operatorname{sign}(s) = 0 \,.$$

$\blacksquare$

Proof of bordism invariance:

Let $j : \partial M \hookrightarrow M$ be the inclusion. Write $H_*(-;\mathbb{R}) =: H_*(-)$.

Consider the commutative diagram:

$$\begin{tikzcd}
H^{2d}(M) \arrow[r, "H^{2d}(j)"] \arrow[d, "\cong", "\_\cap[M,\partial M]"'] \arrow[dr, phantom, "\circlearrowleft"] & H^{2d}(\partial M) \arrow[r, "\delta"] \arrow[d, "\_\cap[\partial M]", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & H^{2d+1}(M,\partial M) \arrow[d, "\_\cap[M,\partial M]", "\cong"'] \\
H_{2d+1}(M,\partial M) \arrow[r, "\partial"] & H_{2d}(\partial M) \arrow[r, "H_{2d}(j)"] & H_{2d}(M)
\end{tikzcd}$$

By the vertical isomorphisms we have $\dim(\underbrace{\operatorname{im} H^{2d}(j)}_{=:L}) = \dim(\ker H_{2d}(j))$. ⊛

- We have
$$\dim H_{2d}(\partial M) = \dim(\ker H_{2d}(j)) + \dim(\operatorname{im} H_{2d}(j))$$
$$\text{UCT}\ \Vert$$
$$\dim H^{2d}(\partial M) = \dim(\ker H_{2d}(j)) + \dim(\operatorname{im}(H_{2d}(j)^*))$$
$$\underset{\text{UCT}}{\overset{⊛}{=}} \; 2 \cdot \dim L$$

↗

- $\langle H^{2d}(j)(x) \cup H^{2d}(j)(y) \, , [\partial M] \rangle$

$$= \langle H^{4d}(j)(x \cup y) \, , [\partial M] \rangle$$

$$= \langle H^{4d}(j)(x \cup y) \, , \partial([M, \partial M]) \rangle$$

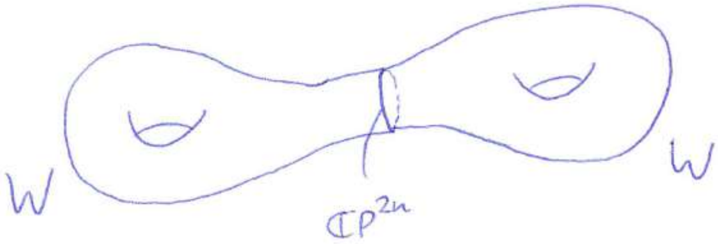
$$= \langle x \cup y \, , \underbrace{H^{4d}(j)}_{\underset{LES}{=} 0} (\partial([M, \partial M])) \rangle$$

$$= 0$$

The claim follows from the lemma. ▨

Corollary: $\mathbb{C}P^{2n}$ is not the boundary of an oriented compact manifold.

Is it the boundary of a compact non-orientable manifold?
No! Assume by contradiction there is such a manifold W.



$D(W) = W \cup_{\partial M} W$ is a $(4n+1)$-dimensional closed manifold.

$$\begin{tikzcd} \mathbb{C}P^{2n} \arrow[r, hook] \arrow[d, hook] & W \arrow[d] \\ W \arrow[r] & D(W) \end{tikzcd}$$

pushout

$$\chi(D(W)) = 2 \cdot \chi(W) - \underbrace{\chi(\mathbb{C}P^{2n})}_{=2n+1} \neq 0$$

↯

... since the Euler characteristic of closed odd-dim. manifolds vanishes. ▨ ⤴

10.6 Intersection numbers

9.2.

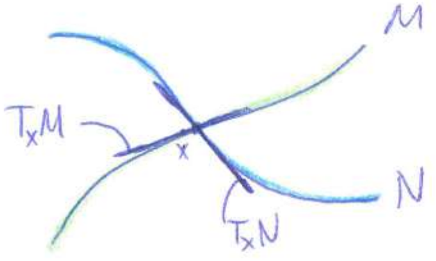
Let W be a closed oriented smooth manifold of dimension w. Let $D : H_i(W) \longrightarrow H^{w-i}(W)$ be the inverse of the Poincare duality isom.

Def.: The intersection product $a \cdot b$ of $a \in H_i(W)$ and $b \in H_j(W)$ is $a \cdot b := P(D(a) \cup D(b)) \in H_{i+j-w}(W)$.

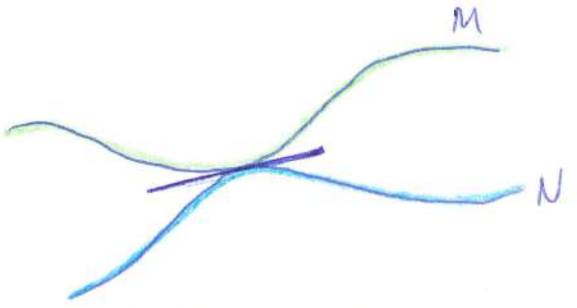
Notation: Let $N \subset W$ be an oriented closed smooth submanifold. The image of $[N] \in H_n(N)$ in $H_n(W)$ is denoted by $[N]_W$.

Def.: Let N, M be smooth submanifolds of W. We say that M and N intersect transversely if

$$T_x M + T_x N = T_x W \quad \forall x \in M \cap N .$$



transversal



not transversal

Theorem: If $N, M \subset W$ intersect transversely and $M \cap N \neq \emptyset$, then $M \cap N \subset W$ is a smooth submanifold of dimension

$$\dim(N) + \dim(M) - \dim(W) .$$

↗

Proof: Let $p \in M \cap N$. Take a chart $h: U \to \mathbb{R}^w$ for W for which M is flat, i.e. $h(U \cap M) = \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^w$.

Consider $q: U \xrightarrow{h} \mathbb{R}^m \times \mathbb{R}^{w-m} \xrightarrow{\text{proj.}} \mathbb{R}^{w-m}$.

Then 0 is a regular value of $q|_{U\cap N}$:

Let $x \in (q|_{U\cap N})^{-1}(0) = q^{-1}(0) \cap U \cap N = U \cap N \cap M$.

$T_x q|_{U\cap N}: T_xN \xrightarrow{T_x(h|_{U\cap N})} \mathbb{R}^m \times \mathbb{R}^{w-m} \longrightarrow \mathbb{R}^{w-m}$ is surjective since

$T_x h: T_xW = T_xN + T_xM \longrightarrow \mathbb{R}^m \times \mathbb{R}^{w-m}$ is an isomorphism and

$T_xh(T_xM) \subseteq \mathbb{R}^m \times \{0\}$.

$\Rightarrow$ $q|_{U\cap N}^{-1}(0) = U \cap N \cap M$ is a smooth submanifold of $U \cap N$.

$\Rightarrow$ $N \cap M$ is a smooth submanifold of N, hence of W. ▨

Def.: Retain the setting of the previous theorem and assume that

$$\dim M + \dim N = \dim W$$

and that M, N and W are oriented.

The intersection number of M and N at $p \in M \cap N$ is defined as

$$\varepsilon_p := \begin{cases} +1 & \text{if } T_pM + T_pN \xrightarrow{\cong} T_pW \text{ is orientation preserving,} \\ -1 & \text{if } T_pM + T_pN \xrightarrow{\cong} T_pW \text{ is orientation reversing.} \end{cases}$$

The intersection number of M and N is then defined as

$$i_W(M,N) = \sum_{p \in M \cap N} \varepsilon_p .$$

⤴

Thm.: (geometric interpretation of $\cup$)

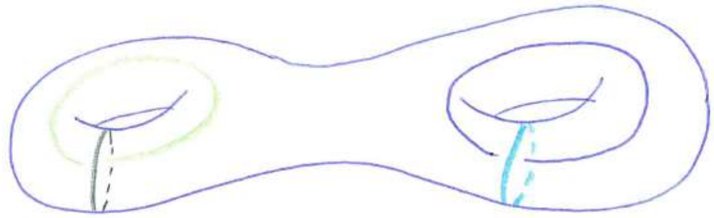
Assume the oriented closed smooth submanifolds $M, N \subset W$ intersect transversely. Then there is an orientation of $M \cap N$ such that

$$[M]_W \cdot [N]_W = [M \cap N]_W .$$

If in addition $\dim M + \dim N = \dim W$, then

$$\langle D([M]_W) \cup D([N]_W) , [W] \rangle = i_W(M,N) .$$

Example:



$$\left.\begin{array}{l} \alpha_1 = D(⬯) \\ \beta_1 = D(⬭) \\ \alpha_2 = D(⬯) \\ \beta_2 = D(⬭) \end{array}\right\} \text{generate } H^1(\Sigma)$$

$\alpha_1 \cup \beta_1$ is a generator of $H^2(\Sigma)$ since ⬭ and ⬯ intersect transversely in exactly one point.

$\alpha_2 \cup \beta_1 = 0$, ...

↗

Example: $\mathbb{RP}^n$ n-dimensional closed manifold

$$H^*(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2[X]/_{(X^{n+1})} \quad , \quad |X| = 1 \quad .$$

In particular, if $x^i = a \in H^i(\mathbb{RP}^n;\mathbb{F}_2)$ and $x^j = b \in H^j(\mathbb{RP}^n;\mathbb{F}_2)$ are $(i+j=n,$ generators, then $< a \cup b \,,\, [\mathbb{RP}^n]_{\mathbb{F}_2} > = 1$.

Generators in H^i and H^j correspond to generators of $\underset{\substack{\rotatebox{90}{$\in$} \\ [\mathbb{RP}^{n-i}]_{\mathbb{RP}^n}}}{H_{n-i}(\mathbb{RP}^n;\mathbb{F}_2)}$ and $\underset{\substack{\rotatebox{90}{$\in$} \\ [\mathbb{RP}^{n-j}]_{\mathbb{RP}^n}}}{H_{n-j}(\mathbb{RP}^n;\mathbb{F}_2)}$ by Poincare duality.

More specifically we take $\mathbb{RP}^{n-i} = \{ [x_0, \dots, x_{n-i}, 0, \dots] \} \subseteq \mathbb{RP}^n$

and $\mathbb{RP}^{n-j} = \{ [0, \dots, 0, y_0, y_1, \dots, y_{n-j}] \} \subseteq \mathbb{RP}^n$.

Note that $\mathbb{RP}^{n-i} \cap \mathbb{RP}^{n-j} = \{ [0, \dots, 0, 1, 0, \dots, 0] \}$ (transversal).

Thm. $\Rightarrow$ $$\Big\langle \underbrace{D([\mathbb{RP}^{n-i}]_{\mathbb{RP}^n})}_{=a} \cup \underbrace{D([\mathbb{RP}^{n-j}]_{\mathbb{RP}^n})}_{=b} \,,\, [\mathbb{RP}^n] \Big\rangle = i_{\mathbb{RP}^n}(\mathbb{RP}^{n-i}, \mathbb{RP}^{n-j})$$
$$= 1 \quad .$$