POWER SERIES OVER THE GROUP RING OF A FREE GROUP AND APPLICATIONS TO NOVIKOV-SHUBIN INVARIANTS

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Abstract. We study power series over the group ring $\mathbb{C}F$ of a free group F. We prove that the von Neumann trace maps rational power series over $\mathbb{C}F$ to algebraic power series. Using the Riemann-Stieltjes formula, we deduce the rationality and positivity of Novikov-Shubin invariants of matrices over $\mathbb{C}F$.

1. Introduction

The motivation for this paper is the following conjecture by J. Lott and W. Lück [13, conjecture 7.1].

Conjecture. *The Novikov-Shubin invariants of the universal covering of a compact Riemannian manifold are positive rational unless they are ∞ or ∞^+.*

We say that the conjecture holds for the group G, if it is true for all compact Riemannian manifolds with fundamental group G. The conjecture for the group G is equivalent to each of the following two conjectures. See [13, conjecture 7.2] and [14, p. 113, proof of 10.5 on p. 371].

Conjecture (Alternative Formulation 1)**.**
The Novikov-Shubin invariants of a finite free G-CW complex are positive rational unless they are ∞ or ∞^+.

Conjecture (Alternative Formulation 2)**.**
The Novikov-Shubin invariant of a matrix $A \in M_n(\mathbb{Z}G)$ over the integral group ring of G is positive rational unless it is ∞ or ∞^+.

The definition of Novikov-Shubin invariants of spaces resp. matrices is reviewed below. The conjecture has been verified for abelian groups by J. Lott [12, proposition 39 on p. 494] (see also [14, p.113]). In this case the

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value ∞ does not occur. So far, no group is known for which the conjecture fails, but computations are rare. For virtually free groups, we settle the conjecture in the positive in 3.6.

Theorem. *The conjecture holds for virtually free groups, and the value ∞ does not occur in that case.*

The general hope is to prove the conjecture for more classes of groups and to provide an algebraic explanation for it in the spirit of Linnell's proof of the Atiyah conjecture [9, 10, 11]. Compared to the progress Linnell and others made concerning the Atiyah conjecture, we are far away from providing an algebraic guideline for proving the conjecture for large classes of groups. However, we link the conjecture to the following algebraic question.

Question. *What is the complexity of the power series lying in the image of the rational power series under the map*

$$\mathrm{tr}_{\mathbb{C}G} : \mathbb{C}G[\![z]\!] \to \mathbb{C}[\![z]\!],\ \sum_{n\geq 0} a_n z^n \mapsto \sum_{n\geq 0} \mathrm{tr}_{\mathcal{N}(G)}(a_n) z^n.$$

Here $\mathrm{tr}_{\mathcal{N}(G)}$ is the von Neumann trace defined below. In 2.19 we prove the following theorem.

Theorem. *For a virtually free group G the map $\mathrm{tr}_{\mathbb{C}G}$ sends rational power series in $\mathbb{C}G[\![z]\!]$ to algebraic power series in $\mathbb{C}[\![z]\!]$.*

This statement reformulates and generalizes older algebraicity results for certain power series, like the one associated to the Markov operator of free groups. See example 2.21. The known methods in this context either consist of ad-hoc combinatorial arguments, use Schützenberger's theorem on the Hadamard product [19, 6], which is related to formal language theory, or employ Voiculescu's theory of free probability [18]. We extend and refine the method in [6], which uses Schützenberger's theorem, to prove the preceding theorem in its generality. Here we benefited from the exposition in [17].

In the next section we compile the necessary facts about noncommutative power series and prove the preceding theorem. In the last section we show how to deduce the rationality and positivity of Novikov-Shubin invariants from that.

Let us now define our main objects of study. Let G be a group. The group von Neumann algebra $\mathcal{N}(G)$ is obtained by completing the left regular representation of $\mathbb{C}G$ on $l^2(G)$ with respect to the weak operator topology. It is equipped with a finite normal faithful trace $\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$. Restricted to $\mathbb{C}G$ the trace is given by $\mathrm{tr}_{\mathcal{N}(G)}(\sum_{g\in G} \lambda_g g) = \lambda_e$. The algebra $M_n(\mathcal{N}(G))$ of $n \times n$-matrices over $\mathcal{N}(G)$ is again a finite von Neumann algebra with trace $\mathrm{tr}_{M_n(\mathcal{N}(G))}((A_{ij})) = \frac{1}{n}\sum_{i=1}^n \mathrm{tr}_{\mathcal{N}(G)}(A_{ii})$.

Consider a self-adjoint operator $A \in \mathcal{A}$ in a finite von Neumann algebra $\mathcal{A}$ with trace $\operatorname{tr}_{\mathcal{A}}$. We associate to it its family of spectral projections $E_\lambda^A \in \mathcal{A}$, $\lambda \in \mathbb{R}$, which are obtained by spectral calculus with respect to the characteristic functions $\chi_{(-\infty,\lambda]}$. The function $F_A(\lambda) = \operatorname{tr}_{\mathcal{A}}(E_\lambda^A)$ is called the spectral density function of A. It is right-continuous. The spectral density function F_A induces a compactly supported Borel probability measure μ_A on $\mathbb{R}$ defined by

$$\mu_A((\lambda,\mu]) = F_A(\mu) - F_A(\lambda).$$

The measure μ_A is called the spectral measure of A. Recall that $\mu_A(\{\lambda\}) \neq 0$ if and only if λ is an eigenvalue of A.

Definition. *For an arbitrary operator $A \in \mathcal{A}$ the Novikov-Shubin invariant $\alpha(A) \in [0,\infty] \cup \{\infty^+\}$ of $A \in \mathcal{A}$ is defined as*

$$\alpha(A) = \begin{cases} \liminf\limits_{\lambda\to 0^+} \dfrac{\ln(F_{A^*A}(\lambda^2) - F_{A^*A}(0))}{\ln(\lambda)} \in [0,\infty] & \textit{if } F_{A^*A}(\lambda^2) > F_{A^*A}(0) \\ & \quad \textit{for } \lambda > 0, \\ \infty^+ & \textit{else.} \end{cases}$$

Note that $F_{A^*A}(\lambda^2) = F_A(\lambda)$ if A is positive. The Novikov-Shubin invariants of a G-CW-complex are defined as the Novikov-Shubin invariants of the differentials of its cellular chain complex. For details see [14].

The results of this article are contained in the author's thesis [15]. I would like to thank Wolfgang Lück for his continuous support and encouragement. Further, I benefited from discussions with Thomas Schick and Warren Dicks.

2. Power Series in Noncommuting Variables

Before we deal with formal power series with noncommuting variables, let us fix the notation in the commutative case.

The ring of formal power series over a (not necessarily commutative) ring R in a set of variables X is denoted by $R[\![X]\!]$. The ring of polynomials is denoted by $R[X]$. Now suppose the coefficient ring k of $k[\![X]\!]$ is a commutative field. An element in $k[\![X]\!]$ is invertible if and only if its constant term is invertible, i.e. non-zero. The quotient field of the integral domain $k[X]$ of polynomials is denoted by $k(X)$. The quotient field of the integral domain $k[\![X]\!]$, denoted by $k(\!(X)\!)$, contains both $k(X)$ and $k[\![X]\!]$. Let D be an integral domain that contains $k(X)$. Then $P \in D$ is said to be *algebraic over* $k(X)$, if there exist $p_0, \ldots, p_d \in k(X)$, not all 0, such that

$$p_d P^d + p_{d-1} P^{d-1} + \cdots + p_0 = 0.$$

By clearing denominators, the p_i above can be assumed to be polynomials, i.e. $p_i \in k[X]$. The algebraic elements in D form a $k(X)$-subalgebra of D.

The power series in $k[\![X]\!] \subset k((X))$, which are algebraic over $k(X)$, are called *algebraic (formal) power series.* We denote the set of all algebraic formal power series by $k_{alg}[\![X]\!]$.

Before we develop analogous notions for the non-commutative setting, we have to recall some concepts from ring theory.

Definition 2.1 (Division Closure and Rational Closure)**.**
Let S be a ring and $R \subset S$ be a subring.

(i) R is *division closed* if for every element in R, which is invertible in S, the inverse already lies in R.
(ii) R is *rationally closed* if for every matrix over R, which is invertible in S, the entries of the inverse already lie in R.
(iii) The *division closure* of R in S denoted by $\mathcal{D}(R \subset S)$ is the smallest division closed subring containing R.
(iv) The *rational closure* of R in S denoted by $\mathcal{R}(R \subset S)$ is the smallest rationally closed subring containing R.

Note that the intersections of division closed resp. rationally closed subrings is division resp. rationally closed. The rational closure has the advantage that it can be explicitly described as follows (see [4, theorem 1.2 on p. 383]).

Theorem 2.2 (Explicit Description of the Rational Closure)**.**
Let $R \subset S$ be a ring extension. Then $s \in S$ is an element of $\mathcal{R}(R \subset S)$ if and only if there is a matrix $A \in M(R)$ over R, which is invertible over S, such that s is an entry of $A^{-1} \in M(S)$. Further, $s \in \mathcal{R}(R \subset S)$ holds if and only if there is a matrix $A \in M_n(R)$, which is invertible over S, and a column vector $b \in R^n$ such that s is a component of the solution u of the matrix equation $Au = b$.

Consider a homomorphism $\phi : S_1 \to S_2$ of ring extensions $R_1 \subset S_1$, $R_2 \subset S_2$, i.e. f restricts to $f_{|R_1} : R_1 \to R_2$. Then ϕ extends canonically to a homomorphism $M(\phi) : M(S_1) \to M(S_2)$, which restricts to the matrix rings of R_1, R_2 respectively. Since $M(\phi)$ maps invertible elements in $M(S_1)$ to invertible elements in $M(S_2)$, the preceding theorem implies the next corollary.

Corollary 2.3. *The rational closure is functorial with respect to homomorphisms of ring extensions, i.e. for a homomorphism $f : S_1 \to S_2$ restricting to the subrings $R_1 \subset S_1$, $R_2 \subset S_2$, we have $\phi(\mathcal{R}(R_1 \subset S_1)) \subset \mathcal{R}(R_2 \subset S_2)$.*

Next we recall the concept of formal power series in noncommuting variables and fix some notation. Let X be a finite set, called an *alphabet*, and let X^* be the free monoid generated by the elements of X. Thus X^* consists of all finite words (strings) $x_1 \dots x_n$ of elements in X including the empty word $1 \in X^*$. The product in X^* is given by concatenation of words. The *length* of $w = x_1 \dots x_n \in X^*$ is given by n, that is the number of letters in w. We write S^+ for $S - \{1\}$ where $S \subset X^*$, and we write X^+ for $X^* - \{1\}$.

A *(formal) power series* in the set of noncommuting variables X over a (not necessarily commutative) ring R is a function $P : X^* \to R$. We write $\langle P, w\rangle$ for $P(w)$, and then use the suggestive notation

$$P = \sum_{w \in X^*} \langle P, w\rangle w.$$

The power series P is called a *polynomial in* X if it has finite support, i.e. $\langle P, w\rangle \neq 0$ holds only for finitely many $w \in X^*$. The set of all formal power series and all polynomials over R in X is denoted by $R\langle\langle X\rangle\rangle$, $R\langle X\rangle$ respectively.

The set of formal power series over R in noncommuting variables has a ring structure (even an R-algebra structure for commutative R). The addition is componentwise and the product is given by

$$\left(\sum_{w \in X^*} a_w w\right) \cdot \left(\sum_{w \in X^*} b_w w\right) = \sum_{w \in X^*} \left(\sum_{uv=w} a_u b_v\right) w.$$

The set of all polynomials $R\langle X\rangle$ is a subring of $R\langle\langle X\rangle\rangle$. A term of the form $a \cdot w$, $a \in R$, $w \in X^*$ is called a *monomial*, and the *degree* of $a \cdot w$ is defined as the length of w.

We say that a sequence $P_1, P_2, \dots$ of formal power series *converges* to $P \in R\langle\langle X\rangle\rangle$ if for every $w \in X^*$ there are only finitely many $i \in \mathbb{N}$ such that $\langle P_i, w\rangle \neq \langle P, w\rangle$. The *augmentation homomorphism* $\epsilon : R\langle\langle X\rangle\rangle \to R$ is the ring map given by $\epsilon(P) = \langle P, 1\rangle$. In the commutative case $\epsilon : R[\![X]\!] \to R$ is defined analogously. A formal power series $P \in R\langle\langle X\rangle\rangle$ is invertible if and only if $\epsilon(P)$ is invertible in R. In this case the inverse is given by the convergent sum

$$P^{-1} = \sum_{k=0}^{\infty} (1 - \epsilon(P)^{-1} \cdot P)^k \cdot \epsilon(P)^{-1}.$$

The analogous statement holds for $R[\![X]\!]$.

Notice that there is a canonical epimorphism $\phi : R\langle\langle X\rangle\rangle \to R[\![X]\!]$ from the formal power series ring in noncommuting variables to the one in commuting variables.

Definition 2.4 (Rational Power Series)**.**
The rational closure $\mathcal{R}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$ is called the *ring of rational power series over* R *in the noncommuting variables* X, and it is denoted by $R_{rat}\langle\langle X\rangle\rangle$. We define $R_{rat}[\![X]\!]$ analogously.

The following theorem is an easy consequence of Gaussian elimination and is certainly well known.

Theorem 2.5. *Let R be an arbitrary ring, and X be a finite set. Then the division closure $\mathcal{D}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$ of the polynomials $R\langle X\rangle$ in the ring of formal power series $R\langle\langle X\rangle\rangle$ coincides with the rational closure $\mathcal{R}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$. The analogous statement for $R[\![X]\!]$ also holds.*

Proof. Let $P \in \mathcal{R}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$. Then, by 2.2, there is a matrix $A \in M_n(R\langle X\rangle)$ $(= M_n(R)\langle X\rangle)$, invertible over $R\langle\langle X\rangle\rangle$, and a vector $b \in (R\langle X\rangle)^n$ such that P is a component of the solution u of the matrix equation $Au = b$. Since A is invertible, the augmentation $\epsilon(A) \in M_n(R)$ must be invertible (over R). So, multiplying with $\epsilon(A)^{-1}$ from the left, we can assume that the matrix equation has the form $(\mathrm{id} + A)u = b$, where A has zero constant term. More generally, consider a system of equations

$$\begin{array}{ccccccccc} (1+A_{11})u_1 & + & A_{22}u_2 & + & \dots & + & A_{1n}u_n & = & b_1 \\ A_{21}u_1 & + & (1+A_{22})u_2 & + & \dots & + & A_{2n}u_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ A_{n1}u_1 & + & A_{n2}u_2 & + & \dots & + & (1+A_{nn})u_n & = & b_n, \end{array}$$

where all A_{ij} lie in $\mathcal{D}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$ and have zero constant term. By induction over n, we show that then there is a unique solution $u = (u_1, \dots, u_n)$ in $\mathcal{D}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$. For $n = 1$ we get $u_1 = (1 + A_{11})^{-1} b_1 \in \mathcal{D}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$. Assume it is true for $n - 1$. Then we can do *Gaussian elimination.* Multiply the first equation on the left with $-A_{j1}(1+A_{11})^{-1} \in \mathcal{D}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$ and add to the jth equation for $2 \le j \le n$. We obtain a system of $n - 1$ equations with the same structure, which has, by induction hypothesis, a unique solution $u_2, \dots, u_n$ lying in the division closure. Solving the first equation, we get $u_1 \in \mathcal{D}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$. The proof for $R[\![X]\!]$ is analogous. □

Remark 2.6. If k is a commutative field, then

$$\mathcal{D}(k[X] \subset k[\![X]\!]) = \mathcal{R}(k[X] \subset k[\![X]\!]) = k(X) \cap k[\![X]\!].$$

The intersection $k(X) \cap k[\![X]\!] \subset k((X))$ consists of the quotients P/Q of polynomials in $k(X)$ such that $Q(0) \neq 0$. On the other hand there is no calculus of fractions for $R_{rat}\langle\langle X\rangle\rangle$. In particular, not every rational series in $R_{rat}\langle\langle X\rangle\rangle$ is a quotient of two polynomials.

Example 2.7. From 2.3 we see that the canonical homomorphism $\phi : R\langle\langle X\rangle\rangle \to R[\![X]\!]$ restricts to a map $R_{rat}\langle\langle X\rangle\rangle \to R_{rat}[\![X]\!]$. On the other hand, it is not true that $\phi(P) \in R_{rat}[\![X]\!]$ implies $P \in R_{rat}\langle\langle X\rangle\rangle$. The formal power series

$$P(x,y) = \sum_{n=0}^{\infty} x^n y^n \in \mathbb{C}\langle\langle x,y\rangle\rangle$$

is not rational (see [17, example 6.6.2 on p. 203]), but its image under ϕ is the geometric series

$$\sum_{n=0}^{\infty} (xy)^n = \frac{1}{1-xy} \in \mathbb{C}[[x,y]].$$

Example 2.8 (Word Problem of $\mathbb{Z}/2 \times \mathbb{Z}/2$)**.**
Let G be a finitely generated group and $S \subset G$ be a finite subset that generates G as a monoid. The *language of the word problem* $\mathcal{W}(G)$ (with respect to S) is defined as the set of words $w = x_1 x_2 \cdots x_n \in S^*$ that reduce to the identity in G. We associate the formal power series

$$P_G = \sum_{w \in \mathcal{W}(G)} w \in \mathbb{Z}\langle\langle S\rangle\rangle.$$

in the noncommuting variables S to $\mathcal{W}(G)$. For instance, we have

$$P_{\mathbb{Z}/2} = (1-x^2)^{-1} = \sum_{n \geq 0} x^{2n},$$

where x represents the generator of $\mathbb{Z}/2$. Now we want to consider the non-trivial example

$$G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \{x, y;\ x^2 = y^2 = 1, x \cdot y = y \cdot x\}$$

with the monoid generators $S = \{x, y\}$. For $z \in \{1, x, y, x \cdot y\} \subset G$ define $\mathcal{W}(z)$ as the set of words in S^* that reduce to z in G. Put $P_z = \sum_{w \in \mathcal{W}(z)} w$. The obvious fact that a word in $\mathcal{W}(z)$ is trivial or ends with x or y leads to the following system of equations for $P_1, P_x, P_y, P_{x \cdot y}$.

$$\begin{aligned} P_1 &= 1 + P_x x + P_y y \\ P_x &= P_1 x + P_{x \cdot y} y \\ P_y &= P_1 y + P_{x \cdot y} x \\ P_{x \cdot y} &= P_x y + P_y x. \end{aligned}$$

Now we could apply noncommutative Gaussian elimination to solve it.

The following definition is due to M. Schützenberger.

Definition 2.9 (Proper Algebraic System)**.**
Let X be an alphabet, and let $Z = \{z_1, \ldots, z_n\}$ be an alphabet disjoint from X. A *proper algebraic system* is a set of equations $z_i = p_i$, $1 \leq i \leq n$ such that

(i) $p_i = p_i(X, Z) \in R\langle X \cup Z\rangle$ for all $1 \leq i \leq n$,

(ii) $\langle p_i, 1\rangle = 0$ and $\langle p_i, z_j\rangle = 0$ for all $1 \leq i, j \leq n$, i.e. p_i has no constant term and no linear terms in the z_j.

A *solution* to the proper algebraic system is an n-tuple $(S_1, \ldots, S_n) \in R\langle\langle X\rangle\rangle^n$ of formal power series in X each having zero constant term and satisfying

$$S_i = p_i(X, Z)_{z_j=S_j} \text{ for } 1 \leq i \leq n.$$

Here $p_i(X, Z)_{z_j=S_j}$ means that we formally substitute each z_j by S_j in $p_i(X, Z)$. Each S_j is called a *component* of the solution.

Definition 2.10 (Algebraic Power Series in Noncommuting Variables)**.** A formal power series $p \in R\langle\langle X\rangle\rangle$ is *algebraic* if $P - \langle P, 1\rangle$ is a component of the solution of some proper algebraic system. The set of all algebraic formal power series in $R\langle\langle X\rangle\rangle$ is denoted by $R_{alg}\langle\langle X\rangle\rangle$.

Theorem 2.11. *Every proper algebraic system has a unique solution.*

The proof is constructive [17, proposition 6.6.3 on p. 203]. The underlying algorithm, called *successive approximation*, will be illustrated in the following example.

Example 2.12. The power series $S = \sum_{n\geq 0} x^n y^n \in \mathbb{Z}\langle\langle x, y\rangle\rangle$ of example 2.7 is algebraic because $S - \langle S, 1\rangle S$ satisfies the equation $z = xy + xzy$. To obtain the solution by the method of successive approximation, we put $S^{(0)} = 0$ and define recursively $S^{(n+1)} = xy + xS^{(n)}y \in \mathbb{Z}\langle x, y\rangle$. The first recursion steps yield

$$\begin{aligned} S^{(0)} &= 0 \\ S^{(1)} &= xy \\ S^{(2)} &= xy + x(xy)y = xy + x^2y^2 \\ S^{(3)} &= xy + x(xy + x^2y^2)y = xy + x^2y^2 + x^3y^3. \end{aligned}$$

The limit $\lim_{n\to\infty} S^{(n)}$ equals $S - \langle S, 1\rangle S$.

The following theorem can be found in [4, theorem 9.17 on p. 135].

Theorem 2.13. *$R_{alg}\langle\langle X\rangle\rangle \subset R\langle\langle X\rangle\rangle$ is a subring containing $R_{rat}\langle\langle X\rangle\rangle$.*

As we would expect from a reasonable notion of *algebraicity* in the noncommutative world, it is compatible with *algebraicity* in the commutative setting. See [17, theorem 6.6.10 on p. 207] and [17, theorem 6.1.12 on p. 168] for the proofs of the next two theorems.

Theorem 2.14. *Let k be a commutative field. Then the algebraic formal power series in noncommuting variables are mapped to algebraic formal power series in commuting variables under the canonical homomorphism $\phi : k\langle\langle X\rangle\rangle \to k[\![X]\!]$.*

Theorem 2.15. *Let k be a commutative field, and let $P \in k(x)((x_1, \ldots, x_n))$ be algebraic over $k(x)(x_1, \ldots, x_n)$. If $P(1, \ldots, 1)$ is a well-defined element in $k((x))$ then $P(1, \ldots, 1) \in k((x))$ is algebraic over $k(x)$.*

Definition 2.16 (Hadamard Product)**.**
Let $P, Q \in R\langle\langle X\rangle\rangle$. The *Hadamard product* $P \odot Q$ of P and Q is defined as

$$P \odot Q = \sum_{w \in X^*} \langle P, w\rangle\langle Q, w\rangle w.$$

The following theorem by Schützenberger [16] is central to the theory of formal power series in noncommuting variables, and it is the crucial ingredient in the proof of our main result 2.19. For a proof see also [17, proposition 6.6.12 on p. 208]. We remark that in [16] rational series are defined using the division closure which is equivalent to our definition by theorem 2.5.

Theorem 2.17 (Schützenberger's Theorem)**.**
Let R be a commutative ring. The Hadamard product of two rational formal power series in $R\langle\langle X\rangle\rangle$ is again rational, and the Hadamard product of an algebraic with a rational formal power series in $R\langle\langle X\rangle\rangle$ is algebraic.

Example 2.18 (Word Problem of Free Groups)**.**
Consider the free group F^n in n letters $x_1, \ldots, x_n$. It is generated as a monoid by $S = \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$. The language of the word problem $\mathcal{W}(F^n)$ is the set of those words in S^* that reduce to the identity under the relations

$$x_i x_i^{-1} = x_i^{-1} x_i = 1, \ 1 \le i \le n.$$

We will construct a proper algebraic system for $P_{F^n} = \sum_{w \in \mathcal{W}(F^n)} w \in \mathbb{Z}\langle\langle S\rangle\rangle$. The algebraicity of P_{F^n} was shown by Chomsky and Schützenberger [3]. For convenience, we give a detailed argument.

We say that an element in $\mathcal{W}(F^n)$ is *atomic* if it cannot be written as the product of two words in $\mathcal{W}(F^n)^+$. For $t \in S$ we define G_t as the subset of $\mathcal{W}(F^n)$ consisting of atomic words whose first letter is t, i.e.

$$G_t = \{w \in \mathcal{W}(F^n); \ w = tv, w \ne uu' \text{ for } u, u' \in \mathcal{W}(F^n)^+\}.$$

Define $P_t = \sum_{w \in G_t} w \in \mathbb{Z}\langle\langle S\rangle\rangle$ and put

$$\bar{t} = \begin{cases} x_i & \text{if } t = x_i^{-1} \\ x_i^{-1} & \text{if } t = x_i. \end{cases}$$

Claim 0: Each word in G_t must end in $\bar{t}$.
For $w \in G_t$ of length ≤ 2 that is certainly true. Assume it is true for words in G_t of length smaller than m where $m > 2$. Let $w \in G_t$ be of length m. Since w reduces to the identity, it contains a substring of the form $x\bar{x}$ with

$x \in S$. This substring is not at the first or last position, otherwise w would not be atomic. So we get that $w = tw_1 x\bar{x} w_2 b$ with $w_1, w_2 \in S^*$, $b \in S$. The word tw_1w_2b is also atomic, and by the induction hypothesis it follows $b = \bar{t}$.

Now define the subset $B_t \subset S^*$ by requiring $G_t = tB_t\bar{t}$, and put $Q_t = \sum_{w \in B_t} w$.

Claim 1: Every $w \in \mathcal{W}(F^n)^+$ can be uniquely written as $w = uv$ with $u \in \mathcal{W}(F^n)$, $v \in G_t$ for some $t \in S$.

For a given $w \in \mathcal{W}(F^n)^+$ define the string v as the string of minimal length in $\mathcal{W}(F^n)^+$ such that there is a factorization $w = uv$. Then there must be a $t \in S$ with $v \in G_t$. The converse of the first claim is trivial:

Claim 2: Every product uv with $u \in \mathcal{W}(F^n)$, $v \in G_t$ lies in $\mathcal{W}(F^n)^+$.

Claim 3: Every $w \in B_t^+$ can be uniquely written as $w = uv$ with $u \in B_t$, $v \in G_q$, $q \neq \bar{t}$.

By the first claim there exist a unique u and $v \in G_q$ with $w = uv$. Because $tw\bar{t} = tuv\bar{t} \in G_t$ is atomic, the string v cannot end with t, hence $q \neq \bar{t}$. It is clear that $tu\bar{t}$ reduces to the identity. If $tu\bar{t}$ would not be atomic then $tw\bar{t} = tuv\bar{t}$ would not be either, and so we must have $u \in B_t$.

Claim 4: Every word $w = uv$ with $u \in B_t$, $v \in G_q$, $q \neq \bar{t}$ lies in B_t^+.

Suppose that $tw\bar{t}$ is not atomic. Then let $tw\bar{t} = u'v'$ be a factorization with $u', v' \in \mathcal{W}(F^n)^+$ and u' having minimal length. We have $u' \in G_t$. Since $tu\bar{t}$ is atomic we must have $u' = tur\bar{t}$. It is $r \in \mathcal{W}(F^n)^+$ because of $q \neq \bar{t}$. So we have $r, r' \in \mathcal{W}(F^n)^+$ with $v = rr'$. This contradicts v being atomic, hence $tw\bar{t} \in G_t$, i.e. $w \in B_t$.

Algebraically, the first and second claim can be expressed by the equation

$$P_{F^n} = 1 + P_{F^n} \sum_{t \in S} P_t, \tag{1}$$

and the third and fourth claim translate into

$$Q_t = 1 + Q_t \sum_{\substack{q \in S \\ q \neq \bar{t}}} P_q \quad \text{for } t \in S. \tag{2}$$

The equations (1), (2) yield the following proper algebraic system with the solution $(P_{F^n}^+, Q_{x_1}^+, Q_{x_1^{-1}}^+, \ldots, Q_{x_n}^+, Q_{x_n^{-1}}^+)$. Here we use the abbreviation $P^+ = P - \langle P, 1 \rangle$ for a power series P.

$$\begin{aligned} P_{F^n}^+ &= (P_{F^n}^+ + 1) \sum_{q \in S} q(Q_q^+ + 1)\bar{q} \\ Q_t^+ &= (Q_t^+ + 1) \sum_{\substack{q \in S \\ q \neq \bar{t}}} q(Q_q^+ + 1)\bar{q} \qquad \text{for } t \in S. \end{aligned}$$

Hence P_{F^n} is algebraic.

Now we want to consider formal power series in one variable over group rings. Let G be a group and R be a ring. The *von Neumann trace on the group ring RG* is the mapping

$$\mathrm{tr}_{RG} : RG \to R, \sum_{g\in G} a_g g \mapsto a_e.$$

For $R = \mathbb{C}$ this is the restriction of $\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$ to $\mathbb{C}G$. The map extends to a map of the associated power series rings (in one variable)

$$\mathrm{tr}_{RG} : RG[\![z]\!] \to R[\![z]\!], \ \sum_{n\geq 0} a_n z^n \mapsto \sum_{n\geq 0} \mathrm{tr}_{RG}(a_n) z^n.$$

Theorem 2.19 (Power Series over the Group Ring of Free Groups)**.**

(i) Let H be a subgroup of G with finite index $n < \infty$, and let k be a commutative ring with $\frac{1}{n} \in k$. Then

$$\mathrm{tr}_{kG}(kG_{rat}[\![z]\!]) \subset \mathrm{tr}_{kH}(kH_{rat}[\![z]\!]).$$

(ii) Let k be a commutative field and F be a virtually free group. Then

$$\mathrm{tr}_{kF}(kF_{rat}[\![z]\!]) \subset k_{alg}[\![z]\!].$$

Proof. (i) By choosing a system of representatives $\{Hg_1, \ldots, Hg_n\} = H\backslash G$ we get an isomorphism of left kH-modules

$$kG \xrightarrow{\cong} \bigoplus_{i=1}^{n} kH, \ g \longmapsto (h_1, \ldots, h_n) \text{ with } h_i = \begin{cases} gg_i^{-1} & \text{if } gg_i^{-1} \in H \\ 0 & \text{else.} \end{cases}$$

This induces the injection ϕ of rings

$$\phi : kG = \hom_{kG}(kG, kG) \hookrightarrow \hom_{kH}(kG, kG) \cong M_n(kH).$$

Let $\Sigma : M_n(kH) \to kH$ be the map defined by taking the sum of the diagonal entries. The canonical extensions of ϕ and Σ to the respective power series rings are denoted by the same symbol. A little computation shows that the von Neumann traces tr_{kG} and tr_{kH} on kG and kH satisfy

$$\frac{1}{n} \mathrm{tr}_{kH} \circ \Sigma \circ \phi = \mathrm{tr}_{kG} .$$

By 2.3 the map ϕ restricts to a homomorphism $\phi : kG_{rat}[\![z]\!] \to M_n(kH)_{rat}[\![z]\!]$. We get the inclusion

$$\mathrm{tr}_{kG} \left(kG_{rat}[\![z]\!]\right) \subset (\mathrm{tr}_{kH} \circ \Sigma)(M_n(kH)_{rat}[\![z]\!]).$$

But from the explicit description of the rational closure (2.2) it is clear that the entries of $M_n(kH)_{rat}[\![z]\!]$ lie in $kH_{rat}[\![z]\!]$. Therefore $\Sigma(M_n(kH)_{rat}[\![z]\!]) \subset kH_{rat}[\![z]\!]$, and the claim follows.

(ii) For the second assertion we can restrict to free groups because of the first part. Furthermore, every free group F is the union of its finitely generated subgroups F_i, $i \in I$. As one knows, the F_i are also free. It is easy to see that $kF_{rat}[\![z]\!]$ is the union of the $(kF_i)_{rat}[\![z]\!]$. So it suffices to deal with finitely generated free groups.

Let F be the free group in n letters $x_1, x_2, \ldots, x_n$, and let S be the alphabet $S = \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$. We denote the empty string in S^* by e. In the sequel we shall frequently use the fact that a formal power series (with commuting or noncommuting variables) is invertible if and only if its augmentation is invertible in the coefficient ring.

By rearranging terms, we get the following ring inclusions.

$$k\langle S\rangle[\![z]\!] \subset k[\![z]\!]\langle\langle S\rangle\rangle \subset k(\!(z)\!)\langle\langle S\rangle\rangle \supset k(z)\langle\langle S\rangle\rangle \supset k(z)\langle S\rangle$$

Thus it makes formally sense to claim

$$(k\langle S\rangle)_{rat}[\![z]\!] \subset (k(z))_{rat}\langle\langle S\rangle\rangle. \tag{3}$$

Let us show this. An element in $(k\langle S\rangle)_{rat}[\![z]\!]$ is an entry in the inverse of some matrix $A \in M_n(k\langle S\rangle[z]) = M_n(k)\langle S\rangle[z]$ which is invertible over $k\langle S\rangle[\![z]\!]$. In particular, the coefficient of z^0 in A is invertible in $M_n(k)\langle S\rangle$. Hence the coefficient of ez^0 of A is invertible in $M_n(k)$. In particular, the coefficient of e, which lies in $M_n(k[z])$, is invertible in $M_n(k[\![z]\!])$, hence invertible in $M_n(k(z))$. Thus A is invertible in $M_n(k(z))\langle\langle S\rangle\rangle = M_n(k(z)\langle\langle S\rangle\rangle)$ implying (3).

The monoid homomorphism $\pi : S^* \to F$ is uniquely defined by $\pi(x_i) = x_i$ and $\pi(x_i^{-1}) = x_i^{-1}$ for $1 \le i \le n$. It extends to a homomorphism $\pi : k\langle S\rangle \to kF$ and then (coefficient-wise) to $\pi : k\langle S\rangle[\![z]\!] \to kF[\![z]\!]$.

Now consider $P \in (kF)_{rat}[\![z]\!]$. The power series P is a component of the solution u of some matrix equation $Au = b$, where $A \in M_n(kF[z]) = M_n(kF)[z]$ is a matrix which is invertible over $kF[\![z]\!]$, and b is a vector in $(kF[z])^n$. Without loss of generality, we can assume that the coefficient of z^0 in A is the identity matrix. Compare the proof of 2.5. Choose a lift $\bar{b}$ of b to $(k\langle S\rangle[z])^n$, i.e. $\pi(\bar{b}) = b$. Obviously, one can choose a lift $\bar{A} \in M_n(k\langle S\rangle[z]) = M_n(k\langle S\rangle)[z]$ of A such that the z^0-coefficient of $\bar{A}$ is the identity matrix. In particular, $\bar{A}$ is invertible in $M_n(k\langle S\rangle[\![z]\!])$. Therefore the respective entry of the solution $\bar{u}$ of the matrix equation $\bar{A}\bar{u} = \bar{b}$ maps to P under π. Thus we have

$$P \in \pi\left((k\langle S\rangle)_{rat}[\![z]\!]\right).$$

Let $\bar{P} \in (k\langle S\rangle)_{rat}[\![z]\!] \overset{(3)}{\subset} (k(z))_{rat}\langle\langle S\rangle\rangle$ be a preimage of P. Denote by $\phi : k(z)\langle\langle S\rangle\rangle \to k(z)[\![S]\!]$ the canonical homomorphism. Let $P_F \in \mathbb{Z}\langle\langle S\rangle\rangle$ be the power series associated to the word problem of F with respect to S. We have seen in 2.18 that P_F is algebraic. Therefore $\bar{P} \odot P_F$ is algebraic, i.e. $\bar{P} \odot P_F \in (k(z))_{alg}\langle\langle S\rangle\rangle$ by 2.17. So $\phi(\bar{P} \odot P_F) \in k(z)[\![S]\!]$ is algebraic by 2.14. Substituting every $s \in S$ by 1, we get a formally well defined power series $\phi(\bar{P} \odot P_F)(1, \ldots, 1) \in k[\![z]\!]$ with

$$\operatorname{tr}_{kF}(P) = \phi(\bar{P} \odot P_F)(1, \ldots, 1) \in k[\![z]\!].$$

Finally, from 2.15 the algebraicity of $\operatorname{tr}_{kF}(P)$ is obtained. □

Example 2.20 (Markov Operator for Free Abelian Groups)**.**
Consider the so-called *Markov operator*

$$M = x + x^{-1} + y + y^{-1} \in \mathbb{C}\mathbb{Z}^2$$

of the free abelian group $\mathbb{Z}^2 = \langle x, y; xy = yx \rangle$. A combinatorial argument shows that the trace $T(z) \in \mathbb{C}[\![z]\!]$ of the rational power series

$$(1 - Mz)^{-1} = \sum_{n=0}^{\infty} M^n z^n \in \mathbb{C}\mathbb{Z}^2[\![z]\!].$$

is given by

$$T(z) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 z^{2n} \in \mathbb{C}[\![z]\!].$$

But this power series can be shown to be not algebraic. Compare [17, 6.3. on p. 217]. We remark that $T(z)$ is *D-finite*, i.e. it satisfies a linear differential equation with polynomial coefficients.

Example 2.21 (Markov Operator for Free Groups)**.**
Now we consider the Markov operator of the free group F^k of rank k in the letters x_i, $1 \leq i \leq k$.

$$M = \sum_{i=1}^{k} x_i + x_i^{-1} \in \mathbb{C}F^k$$

Let $X = \{x_1, \ldots, x_k, x_1^{-1}, \ldots, x_k^{-1}\}$. We compute the trace $T = T(z) \in \mathbb{C}[\![z]\!]$ of the formal power series

$$(1 - Mz)^{-1} = \sum_{i=0}^{\infty} M^i z^i \in \mathbb{C}F^k[\![z]\!].$$

This problem was studied and solved by a large number of people. One way to solve it is to apply Voiculescu's machinery of free probability (see [18, p. 28]). The algebraicity of T is also shown in [19]. The following argument uses 2.18. We begin with a general remark.

Let Σ an alphabet. For $w \in \Sigma^*$ denote by $|w|$ the length of w. The map

$$\psi : \mathbb{C}\langle\langle\Sigma\rangle\rangle \longrightarrow \mathbb{C}[\![z]\!]$$
$$\sum_{w\in\Sigma^*} a_w w \mapsto \sum_{n\geq 0}\left(\sum_{|w|=n} a_w\right) z^n$$

is a ring homomorphism. Note that in the notation of 2.18 we have $T = \psi(P_{F^k})$. By symmetry, we have that $S := \psi(P_t)$ is the same for all $t \in X$. Because of $P_t = tQ_t\bar{t}$ we obtain $\psi(Q_t) = z^{-2}S$. The equations (1) and (2) in example 2.18 yield the following system of equations after applying ψ.

$$T = 1 + 2kTS$$
$$z^{-2}S = 1 + z^{-2}(2k-1)S^2$$

The solution T of this system satisfies the algebraic equation

$$(1 - 4k^2z^2)T(z)^2 + (2k-2)T(z) - (2k-1) = 0.$$

3. The Spectrum of Matrices over the Group Ring of a Free Group

In this section we study spectra of operators in a finite von Neumann algebra. More precisely, we compute the spectral density functions of self-adjoint operators. Applying the results of the last section, we will show that the Novikov-Shubin invariants of operators in $M_n(\mathbb{C}F) \subset M_n(\mathcal{N}(F))$, where F is a virtually free group, are positive and rational unless they are ∞^+.

The idea is to consider a power series built from the operator, which codifies all its spectral information. Using the Riemann-Stieltjes inversion formula (a well-known tool in this context), we then extract all the spectral information, we need, from this power series. This method is also employed in [18] to compute spectra of operators.

Definition 3.1 (Cauchy Transform)**.**
The *Cauchy transform* G_μ of a finite, compactly supported Borel measure μ on $\mathbb{R}$ is defined as the function on $\mathbb{C}^+ = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$ given by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}.$$

In our context the measure will be the spectral measure μ_A of a self-adjoint operator A in a finite von Neumann algebra $\mathcal{A}$.

Lemma 3.2 (Cauchy Transform of the Spectral Measure)**.**
We have the following equality of holomorphic functions on $\{z \in \mathbb{C}^+; \|z\| > \|A\|\}$.

$$G_{\mu_A}(z) = \operatorname{tr}_{\mathcal{A}}((z-A)^{-1}).$$

Remark 3.3. In the sequel $\int_a^b$ means $\int_{[a,b]}$, and $\int_a^{b^-}$ stands for $\int_{[a,b)}$ etc.

Proof. The support of μ lies in the spectrum of A, in particular in $[-\|A\|, \|A\|]$. For $z > \|A\|$ the operator $z - A$ is invertible. We get for $\|z\| > \|A\|$

$$G_{\mu_A}(z) = \int_{-\|A\|}^{\|A\|} \frac{d\mu_A(t)}{z-t} = \int_{-\|A\|}^{\|A\|} \left(\sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}} \right) d\mu_A(t) = \sum_{n=0}^{\infty} \left(\int_{-\|A\|}^{\|A\|} \frac{t^n}{z^{n+1}} d\mu_A(t) \right)$$
$$= \sum_{n \geq 0} \operatorname{tr}_{\mathcal{A}}(A^n z^{-n-1})$$
$$= \operatorname{tr}_{\mathcal{A}}((z-A)^{-1}).$$

Here recall that $\sum_{n\geq 0} A^n z^{-n-1}$ converges to $(z-A)^{-1}$ in the norm topology [8, lemma 3.1.5 on p. 175], and that $\operatorname{tr}_{\mathcal{A}}$ is continuous with respect to the ultraweak topology. □

Theorem 3.4 (Riemann-Stieltjes Inversion Formula)**.**
Let μ be a finite, compactly supported Borel measure on $\mathbb{R}$. Let $a, b \in \mathbb{R}$ such that $\mu(\{a\}) = \mu(\{b\}) = 0$. Then

$$\mu([a,b]) = \lim_{y\to 0^+} \left(-\frac{1}{\pi} \int_a^b \operatorname{Im} G_\mu(x+iy) dx \right).$$

Proof. It is a well-known fact, known as the Riemann-Stieltjes Inversion Formula, that μ is the weak limit of the measures $-\frac{1}{\pi} \operatorname{Im} G_\mu(x+iy)dx$ [7, p. 92-93]. By [2, Satz 30.12 on p. 228] this yields the statement provided $\mu(\{a\}) = \mu(\{b\}) = 0$. □

Since we could not find a reference for the following lemma, we include its proof for the convenience of the reader.

Lemma 3.5. *Let μ be a finite, compactly supported Borel measure on $\mathbb{R}$. If $G_\mu(z)$ has a holomorphic extension around $t_0 \in \mathbb{R}$, then $\mu(\{t_0\}) = 0$ holds.*

Proof. Write μ as $\mu = \alpha \cdot \delta_{t_0} + \mu_0$, $\alpha \in \mathbb{R}_{\geq 0}$, where δ_{t_0} is the Dirac measure concentrated at t_0, and the measure μ_0 satisfies $\mu_0(\{t_0\}) = 0$. Then we get

$$G_\mu(z) = \alpha \cdot \frac{1}{z - t_0} + \int_{\mathbb{R}} \frac{d\mu_0(t)}{z-t}.$$

Because $G_\mu(z)$ has an analytic extension around t_0, we have in particular

$$\lim_{y\to 0^+} iy \cdot G_\mu(t_0 + iy) = 0.$$

Next we show that

$$\lim_{y\to 0^+}\int_{\mathbb{R}} \frac{iy}{(t_0+iy)-t}\,d\mu_0(t) = 0. \tag{4}$$

This would imply $\alpha = 0$ and finish the proof. The absolutes values of the summands on the right side in

$$\frac{iy}{(t_0+iy)-t} = \frac{y^2}{(t_0-t)^2+y^2} + i\frac{y(t_0-t)}{(t_0-t)^2+y^2}$$

are ≤ 1 for all $y \neq 0$ and $t \in \mathbb{R}$. Because of σ-additivity and the finiteness of μ_0 we have $\lim_{\epsilon\to 0^+}\mu_0([t_0-\epsilon, t_0+\epsilon]) = \mu_0(\{t_0\}) = 0$. For $n \in \mathbb{N}$ choose $\epsilon > 0$ such that $\mu_0([t_0-\epsilon, t_0+\epsilon]) < \frac{1}{2n}$ holds. Set $\mathbb{R}(\epsilon) = \mathbb{R} - [t_0-\epsilon, t_0+\epsilon]$. By the majorized convergence theorem we obtain

$$\lim_{y\to 0^+}\int_{\mathbb{R}(\epsilon)} \frac{iy}{(t_0+iy)-t}\,d\mu_0(t) =$$

$$\int_{\mathbb{R}(\epsilon)} \lim_{y\to 0^+}\frac{y^2}{(t_0-t)^2+y^2}\,d\mu_0(t) + \int_{\mathbb{R}(\epsilon)} \lim_{y\to 0^+}\frac{y(t_0-t)}{(t_0-t)^2+y^2}\,d\mu_0(t) = 0.$$

For every $y > 0$ we get the estimate

$$\left|\int_{t_0-\epsilon}^{t_0+\epsilon} \frac{iy}{(t_0+iy)-t}\,d\mu_0(t)\right| \leq$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon}\left|\frac{y^2}{(t_0-t)^2+y^2}\right| d\mu_0(t) + \int_{t_0-\epsilon}^{t_0+\epsilon}\left|\frac{y(t_0-t)}{(t_0-t)^2+y^2}\right| d\mu_0(t) \leq \frac{1}{n}$$

Hence the limit in (4) is bounded above by $\frac{1}{n}$. Because $n \in \mathbb{N}$ was chosen arbitrarily, (4) follows. □

Theorem 3.6 (Rationality and Positivity)**.**
Let F be a virtually free group and $\mathbb{Q} \subset k \subset \mathbb{C}$ be a field. Let $A \in M_n(kF) \subset M_n(\mathcal{N}(F))$ be a self-adjoint operator in the finite von Neumann algebra $M_n(\mathcal{N}(F))$, which lives over the group ring. Then the following holds.

(i) The Novikov-Shubin invariant $\alpha(A)$ is positive rational unless it is ∞^+.
(ii) The operator A has a finite number of eigenvalues, and they lie in the algebraic closure of k.
(iii) The spectral density function F_A is piecewise smooth.

Proof. The entries of $z(1-Az)^{-1} \in M_n(kF)[\![z]\!]$ lie in the rational closure $(kF)_{rat}[\![z]\!]$. Due to theorem 2.19, the formal power series

$$q(z) = \operatorname{tr}_{M_n(kF)}\big(z(1-Az)^{-1}\big) = \sum_{i=1}^{n} \operatorname{tr}_{kF}\big((z(1-Az)^{-1})_{ii}\big)$$

is algebraic over $k(z)$. From a non-trivial algebraic equation of $q(z)$ it is obvious that $q(z^{-1})$ also satisfies a non-trivial algebraic equation over $k(z)$. In the domain $\{z \in \mathbb{C}^+; \|z\| > \|A\|\}$ the function $q(z^{-1})$ is convergent, and we have $G_{\mu_A}(z) = q(z^{-1})$, due to 3.2. Therefore there is a non-constant polynomial $P(w,z) = p_n(z)w^n + \cdot + p_0(z)w^0 \in k[w,z]$, $p_i(z) \in k[z]$, $p_n \neq 0$ such that

$$P(G_{\mu_A}(z), z) = 0$$

holds in $\{z \in \mathbb{C}^+; \|z\| > \|A\|\}$ – thus in every domain $G_{\mu_A}(z)$ can be analytically extended to. We can assume that $P(w,z)$ is irreducible (compare [1, p. 293]). Let $Z \subset \mathbb{C}$ be the finite set consisting of the zeroes of p_n and the zeroes of the discriminant of P. We remind the reader that the discriminant of P is a polynomial over k, and the zeroes of the discriminant are exactly the points z_0 such that $Q(w) = P(w, z_0)$ has multiple roots. In particular, Z lies in the algebraic closure of k. From the domain $\{z \in \mathbb{C}^+; \|z\| > \|A\|\}$ the function $G_{\mu_A}(z)$ can be analytically extended along any arc which does not pass through a point of Z [1, p. 294]. Equivalently, $G_{\mu_A}(z)$ can be analytically extended to every simply connected domain not containing Z.

For an eigenvalue λ of A we have $\mu_A(\{\lambda\}) > 0$, and 3.5 implies that the eigenvalues lie in Z. Thus they are contained in the algebraic closure of k.

Let $\lambda \in \mathbb{R} - Z$. There is an open ball U around λ such that G_{μ_A} can be analytically extended to U. In particular, for $\epsilon \in U \cap \mathbb{R}$ we have $\mu_A(\{\epsilon\}) = 0$ by 3.5, and the Riemann-Stieltjes inversion formula yields

$$F_A(\lambda) - F_A(\epsilon) = \lim_{y \to 0^+} \left(-\frac{1}{\pi} \cdot \int_{\epsilon}^{\lambda} \operatorname{Im} G_{\mu_A}(x+iy)\,dx \right).$$

Now the majorized convergence theorem implies

$$F_A(\lambda) - F_A(\epsilon) = -\frac{1}{\pi} \cdot \int_{\epsilon}^{\lambda} \operatorname{Im} G_{\mu_A}(x)\,dx.$$

Thus $F_A(\lambda)$ is smooth outside of Z, and the derivative there is

$$F'_A(\lambda) = -\frac{1}{\pi} \cdot \operatorname{Im} G_{\mu_A}(\lambda). \tag{5}$$

Next we show $\alpha(A) \in \mathbb{Q}_{>0} \cup \{\infty^+\}$. The Novikov-Shubin invariant is defined using the spectral density function of A^*A, so we can assume that

A is positive. Because G_{μ_A} is algebraic there exists $k \in \mathbb{N}$ such that $G_{\mu_A}(z^k)$ can be analytically extended to a pointed neighborhood of 0 having 0 as a pole (see [1, theorem 4 on p. 297]). Therefore $G_{\mu_A}(z^k)$ has an expansion as a Laurent series with finitely many terms of negative exponent. Put $S(\lambda) = F_A(\lambda^k)$. From $S'(\lambda) = -\frac{k}{\pi} \cdot \operatorname{Im} G_{\mu_A}(\lambda^k)\lambda^{k-1}$ for small $\lambda > 0$ we see, by integrating, that $S(\lambda)$ has the form

$$S(\lambda) - S(\epsilon) = \sum_{i=N}^{\infty} c_i \lambda^i + c\ln(\lambda) - \sum_{i=N}^{\infty} c_i \epsilon^i - c\ln(\epsilon) \tag{6}$$

with $N \in \mathbb{Z}$, $c, c_i \in \mathbb{R}$ and $0 < \epsilon \leq \lambda$ small enough. For fixed λ and $\epsilon \to 0^+$ (6) stays bounded because the spectral density function is bounded. In particular, we get $\lim_{\epsilon \to 0^+} \sum_{i=N}^{\infty} c_i \epsilon^{i+1} = 0$ because of $\lim_{\epsilon \to 0^+} \epsilon \ln(\epsilon) = 0$. This implies $c_i = 0$ for $i < 0$, and so c must be zero for (6) to stay bounded. Using the fact that $S(\lambda) = F_A(\lambda^k)$ is right-continuous, we finally get

$$S(\lambda) - S(0) = \sum_{i=M}^{\infty} c_i \lambda^i$$

with $M > 0$. If all c_i are zero, then $F_A(\lambda) - F_A(0)$ is constant for small λ, and then $\alpha(A) = \infty^+$ follows. Now consider the case that not all c_i are zero. Without loss of generality, we assume that $c_M \neq 0$. By the l'Hospital rule we get

$$\lim_{\lambda \to 0^+} \frac{\ln(F_A(\lambda^k) - F_A(0))}{\ln(\lambda)} = \lim_{\lambda \to 0^+} \frac{\ln(S(\lambda) - S(0))}{\ln(\lambda)} = M.$$

Therefore we obtain

$$\alpha(A) = \lim_{\lambda \to 0^+} \frac{\ln(F_A(\lambda) - F_A(0))}{\ln(\lambda)} = \frac{M}{k} \in \mathbb{Q}_{>0}.$$

□

Remark 3.7. In general, it is not clear that the limes inferior in the definition of $\alpha(A)$ can be replaced by a limit. The previous proof shows that it is possible for operators over the group ring of a virtually free group. We say that these operators have the *limit property*.

Part (ii) of 3.6, i.e. the algebraicity of the eigenvalues, is shown in [5] by a different method. There it is proven not only for virtually free but for all ordered groups satisfying the strong Atiyah conjecture over the complex group ring.

Example 3.8. Using the Riemann-Stieltjes inversion formula we are able to compute an explicit formula for the spectral measure μ_A of the Markov operator $A = x + x^{-1} + y + y^{-1}$ of the free group $\mathbb{Z} * \mathbb{Z} = \langle x, y\rangle$. The power series $T(z) = \sum_{n\geq 0} \mathrm{tr}_{\mathcal{N}(\mathbb{Z}*\mathbb{Z})}(A^n)z^n$ satisfies the equation (see 2.21)

$$(1 - 16z^2)T(z)^2 + 2T(z) - 3 = 0.$$

The explicit solution to this equation is

$$T(z) = \frac{3}{1 \pm 2\sqrt{1 - 12z^2}}.$$

Because of $G_{\mu_A}(z) = z^{-1}T(z^{-1})$ we obtain

$$G_{\mu_A}(z) = \frac{3}{z \pm 2\sqrt{z^2 - 12}} = \frac{3(z \mp 2\sqrt{z^2 - 12})}{z^2 - 4(z^2 - 12)}.$$

There are the ("boundary") conditions $F'_A(\lambda) \geq 0$ and $F'_A(\lambda) = 0$ outside a compact set. With this in mind, equation (5) implies

$$F'_A(\lambda) = \begin{cases} \frac{6\sqrt{12-\lambda^2}}{\pi(48-3\lambda^2)} & \text{if } |\lambda| < \sqrt{12} \\ 0 & \text{if } |\lambda| > \sqrt{12}. \end{cases}$$

Thus the support of μ_A is $[-\sqrt{12}, \sqrt{12}]$, and in that interval we have the equality of measures

$$\mu_A = \frac{6\sqrt{12 - \lambda^2}}{\pi(48 - 3\lambda^2)} d\lambda.$$

Integrating yields for $|\lambda| < \sqrt{12}$

$$F_A(\lambda) - F_A(0) = \frac{2}{\pi} \arcsin\left(\frac{\lambda}{2\sqrt{3}}\right) + \frac{1}{2\pi} \arctan\left(\frac{2(3 - \lambda)}{\sqrt{12 - \lambda^2}}\right) + \frac{1}{2\pi} \arctan\left(\frac{2(-3 - \lambda)}{\sqrt{12 - \lambda^2}}\right).$$

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