ℓ^2-Betti numbers and their approximation by finite-dimensional analogues

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Betti numbers

Homology $H_i(X;\mathbb{C})$ and $\mathbb{C}$-dimension: $\beta_i(X) = \dim_{\mathbb{C}} H_i(X;\mathbb{C})$.

Attempt at equivariant Betti numbers

Let $\Gamma = \pi_1(X)$. Then $H_i(\widetilde{X};\mathbb{C})$ is a module over the **group ring**

$$\mathbb{C}[\Gamma] = \Big\{\sum_{\gamma\in\Gamma} a_\gamma \gamma \mid \text{finite sum, } a_\gamma \in \mathbb{C}\Big\}.$$

Pick a nice dimension of $\mathbb{C}[\Gamma]$-modules and consider $\dim_{\mathbb{C}[\Gamma]} H_i(\widetilde{X};\mathbb{C})$.

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Problem

Such $\dim_{\mathbb{C}[\Gamma]}$ might not exist: For $\Gamma = F_2$ the differential

$$C_1(\widetilde{S^1 \vee S^1};\mathbb{C}) = \mathbb{C}[\Gamma]^2 \hookrightarrow \mathbb{C}[\Gamma] = C_0(\widetilde{S^1 \vee S^1};\mathbb{C})$$

is injective. Hence you cannot have additivity of $\dim_{\mathbb{C}[\Gamma]}$.

ℓ^2-Betti numbers try to remedy this situation!

Group von Neumann algebra

$$\mathbb{C}[\Gamma] \subset \ell^2(\Gamma) = \{\sum a_\gamma \gamma \mid \sum |a_\gamma|^2 < \infty\}$$
$$L(\Gamma) = \{T \colon \ell^2(\Gamma) \to \ell^2(\Gamma) \text{ bounded } \mid \forall_{\gamma\in\Gamma}\ T(\gamma x) = \gamma T(x)\}$$

$\mathbb{C}[\Gamma]$ embeds (densely) into $L(\Gamma)$ as right multiplication operators.

Finite trace

$$\begin{array}{ccc} \mathbb{C}[\Gamma] & \longrightarrow & L(\Gamma) \\ & {\scriptstyle \sum a_\gamma \gamma \mapsto a_e} \searrow & \downarrow {\scriptstyle T \mapsto \mathrm{tr}_\Gamma(T) = \langle T(e), e\rangle} \\ & & \mathbb{C} \end{array}$$

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Matrix extension for $T = (T_{ij})$:

$$\mathrm{tr}_\Gamma(\ell^2(\Gamma)^n \xrightarrow{T} \ell^2(\Gamma)^n) := \sum_i \mathrm{tr}_\Gamma(T_{ii})$$

Trace property: $\mathrm{tr}_\Gamma(ST) = \mathrm{tr}_\Gamma(TS)$

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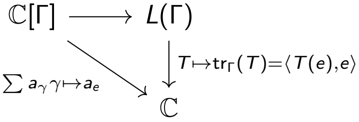

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von Neumann Dimension

$$\dim_\Gamma(A) := \operatorname{tr}_\Gamma\big(\operatorname{pr}_A\colon \ell^2(\Gamma)^n \to A \subset \ell^2(\Gamma)^n\big)$$

for a closed Γ-invariant subspace A.

Equivariant CW-complexes

We consider CW-complexes with cellular actions. The **cellular chain complex** $C_*(X)$ of a (free) Γ-CW-complex is a (free) $\mathbb{Z}[\Gamma]$-chain complex.

ℓ^2-Betti numbers (Atiyah, Dodziuk)

Let X be a free Γ-CW complex with cocompact skeleta.

$$\beta_n^{(2)}(X;\Gamma) = \dim_\Gamma\big(\bar{H}^n(\hom_{\mathbb{Z}[\Gamma]}(C_*(X), \ell^2(\Gamma))\big) \qquad \textbf{reduced} \text{ cohomology!}$$

$$\beta_n^{(2)}(M) = \beta_n^{(2)}(\widetilde{M}; \pi_1(M))$$

$$\beta_n^{(2)}(\Gamma) = \beta_n^{(2)}(E\Gamma;\Gamma)$$

Here $E\Gamma$ is a **classifying space** of Γ, that is, $E\Gamma \simeq *$ and $\Gamma \curvearrowright E\Gamma$ freely.

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An example

Write $\Gamma = \pi_1(S^1) = \mathbb{Z} = \langle t\rangle$. Then $\beta_i^{(2)}(S^1) = 0$.

$$\hom_{\mathbb{Z}[\mathbb{Z}]}(C_*(\widetilde{S^1}),\ell^2(\mathbb{Z})) \cong \big(\ell^2(\mathbb{Z}) \xrightarrow{\_\cdot(t-1)} \ell^2(\mathbb{Z})\big)$$

Basic properties

- equivariant homotopy invariants
- Euler-Poincare formula
- Künneth formula
- Poincare duality

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ℓ^2-Betti numbers and Euler characteristic

Number of n-cells in $\Gamma\backslash X =$

$$\dim_\Gamma \underbrace{\left(\hom_{\mathbb{Z}[\Gamma]}(C_n(X), \ell^2(\Gamma))\right)}_{=: C^n_{(2)}}$$

$$Z^i \hookrightarrow C^i_{(2)} \overset{\text{weak}}{\twoheadrightarrow} \bar{B}^{i+1}$$

$$\bar{B}^i \hookrightarrow Z^i \twoheadrightarrow \bar{H}^i$$

$$\begin{aligned}
\chi(\Gamma\backslash X) = \sum_i (-1)^i \dim_\Gamma(C^i_{(2)}) &= \sum_i (-1)^i (\dim_\Gamma(Z^i) + \dim_\Gamma(\bar{B}^{i+1})) \\
&= \sum_i (-1)^i (\dim_\Gamma(\bar{B}^i) + \dim_\Gamma(\bar{H}^i) + \dim_\Gamma(\bar{B}^{i+1})) \\
&= \sum_i (-1)^i \beta_i^{(2)}(X; \Gamma)
\end{aligned}$$

Some theorems

- $\Lambda, \Gamma < G$ lattices $\Rightarrow \beta_i^{(2)}(\Gamma)\,\mathrm{covol}(\Lambda) = \beta_i^{(2)}(\Lambda)\,\mathrm{covol}(\Gamma)$. (Gaboriau)
- $\beta_i^{(2)}(\Gamma) = 0$ for infinite amenable Γ. (Cheeger-Gromov)
- Vanishing of $\beta_i^{(2)}(\Gamma)$ is QI-invariant. (Pansu)

Two conjectures

- The ℓ^2-Betti numbers of a finite CW complex with torsionfree fundamental groups are integers. (**Atiyah conjecture**)
- The ℓ^2-Betti numbers of a closed aspherical manifold are concentrated in the middle dimension (**Singer conjecture**)

Atiyah vs. Singer

The Singer conjecture is about ℓ^2-Betti numbers of groups whereas the Atiyah conjecture is about $\mathbb{C}[\Gamma]$-modules and their Γ-dimension.

Kaplansky's conjectures

Direct finiteness (conjecture). $ab = 1$ in $\mathbb{C}[\Gamma]$ implies $ba = 1$.

Assume that Γ is torsionfree.

Idempotent conjecture. $p^2 = p$ in $\mathbb{C}[\Gamma]$ implies $p \in \{0, 1\}$.

Zero divisor conjecture. $ab = 0$ in $\mathbb{C}[\Gamma]$ implies $a = 0$ or $b = 0$.

The same statements are conjectured for $\mathbb{F}_p[\Gamma]$. In that case direct finiteness is known for sofic groups (Elek-Szabo).

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Some methods

- ℓ^2-methods
- Finite-dimensional approximation
- Localization (later)

The approximation and localization methods are also available for $\mathbb{F}_p[\Gamma]$.

Zero divisor conjecture by ℓ^2

ZDC is implied by the **Atiyah conjecture** which translates into:

$$\dim_\Gamma(\ker(r_a)) \in \mathbb{N} \text{ for every } a \in \mathbb{C}[\Gamma].$$

$$\begin{aligned} ab = 0 \text{ and } a \neq 0 &\Rightarrow \dim_\Gamma(\ker(r_b\colon \ell^2(\Gamma) \to \ell^2(\Gamma))) > 0 \\ &\Rightarrow \dim_\Gamma(\ker(r_b\colon \ell^2(\Gamma) \to \ell^2(\Gamma))) = 1 \\ &\Rightarrow b = 0 \end{aligned}$$

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Direct finiteness by ℓ^2

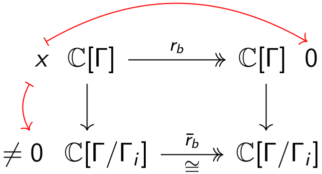
$$K \hookrightarrow \ell^2(\Gamma) \underset{r_b}{\overset{\overset{r_a}{\longleftarrow}}{\twoheadrightarrow}} \ell^2(\Gamma)$$

$$\begin{aligned} \dim_\Gamma(\ell^2(\Gamma)) = \dim_\Gamma(K) + \dim_\Gamma(K^\perp) &= \dim_\Gamma(K) + \dim_\Gamma(\ell^2(\Gamma)) \\ &\Rightarrow \dim_\Gamma(K) = 0 \Rightarrow K = 0. \end{aligned}$$

Direct finiteness by approximation

Let Γ be **residually finite**:

$$\begin{cases} \Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 \dots \\ \Gamma_i < \Gamma \text{ normal and finite index} \\ \cap \Gamma_i = \{e\} \end{cases}$$



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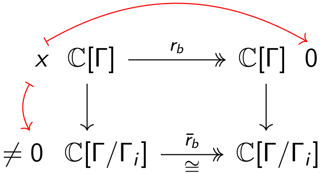

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Lück's approximation theorem

$$\dim_\Gamma \ker(r_a \colon \ell^2(\Gamma) \to \ell^2(\Gamma)) = \lim_{i \to \infty} \frac{\dim_{\mathbb{C}} \ker(\overline{r_a} \colon \mathbb{C}[\Gamma/\Gamma_i] \to \mathbb{C}[\Gamma/\Gamma_i])}{[\Gamma : \Gamma_i]}$$

for $a \in \mathbb{Z}[\Gamma]$

Approximation theorem (Lück)

Let $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ be a residual chain. Let X be a finite free Γ-CW complex. Then

$$\beta_n^{(2)}(X;\Gamma) = \lim_{i\to\infty} \frac{b_n(\Gamma_i\backslash X)}{[\Gamma : \Gamma_i]}$$

Version for universal coverings

Let M be a finite CW complex and $\pi_1(M) = \Gamma_0 > \Gamma_1 > \dots$ be a residual chain. Let $M_i \to M$ be the covering associated to Γ_i. Then

$$\beta_n^{(2)}(M) = \lim_{i\to\infty} \frac{b_n(M_i)}{[\Gamma : \Gamma_i]}$$

Version for groups only

Let $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ be a residual chain. Assume that Γ admits a finite type classifying space. Then

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Version for spaces

Let $\Gamma = \Gamma_0 > \Gamma_1 > \ldots$ be a residual chain. Let X be a finite free Γ-CW complex. Then

$$\beta_n^{(2)}(X;\Gamma) = \lim_{i\to\infty} \frac{b_n(\Gamma_i\backslash X)}{[\Gamma : \Gamma_i]}$$

Version for group rings

Let $\Gamma = \Gamma_0 > \Gamma_1 > \ldots$ be a residual chain. Then

$$\dim_\Gamma(r_A\colon \ell^2(\Gamma)^d \to \ell^2(\Gamma)^d) = \lim_{i\to\infty} \frac{\dim_{\mathbb{C}} \ker(\overline{r_A}\colon \mathbb{C}[\Gamma/\Gamma_i]^d \to \mathbb{C}[\Gamma/\Gamma_i]^d)}{[\Gamma : \Gamma_i]}$$

for every matrix $A \in M_d(\mathbb{Z}[\Gamma])$.

Comparing chain complexes

Suppose X has d equivariant n-cells. Then

$$C^n_{(2)} := \hom_{\mathbb{Z}[\Gamma]}(C_n(X), \ell^2(\Gamma)) \cong \ell^2(\Gamma)^d$$
$$\hom_{\mathbb{Z}}(C_n(\Gamma\backslash X), \mathbb{C})) \cong \ell^2(\Gamma/\Gamma_i)^d = \mathbb{C}[\Gamma/\Gamma_i]^d.$$

The differentials in the second chain complex are the reductions of the ones in the first.

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The Laplacian

$$\Delta^n := (d^n)^* \circ d^n + d^{n-1} \circ (d^{n-1})^* \colon C^n_{(2)} \to C^n_{(2)}$$

- If d^n is given by multiplication with $A \in M_{d,d'}(\mathbb{Z}[\Gamma])$, then $(d^n)^*$ is given by multiplication with $A^* \in M_{d',d}(\mathbb{Z}[\Gamma])$ obtained by transposition and replacing in each entry γ by γ^{-1}.
- **Easy fact**: $\ker(\Delta^n) \to \bar{H}^n(C^*_{(2)})$ is an isomorphism.

Let $A\colon \ell^2(\Gamma) \to \ell^2(\Gamma)$ be a positive Γ-equivariant operator.

Spectral calculus

$$\mathrm{Poly}([0, \|A\|]) \to L(\Gamma), \quad p \mapsto p(A)$$

extends to bounded Borel functions on $[0, \|A\|]$.

Spectral measure

Riesz representation theorem $\Rightarrow \exists$ Borel probability measure μ supported on $[0, \|A\|]$:

$$\int_{\mathbb{R}} f d\mu = \mathrm{tr}_\Gamma(f(A)).$$

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At zero

$$\chi_{\{0\}}(A) = \mathrm{pr}_{\ker(A)} \quad \mu(\{0\}) = \mathrm{tr}_\Gamma(\mathrm{pr}_{\ker(A)}) = \dim_\Gamma(\ker(A))$$

The case of finite Γ

$$|\Gamma| \,\mathrm{tr}_\Gamma(\mathrm{pr}_{\ker(A)}) = |\Gamma| \langle \mathrm{pr}_{\ker(A)}(e), e\rangle = \sum_{\gamma \in \Gamma} \langle \mathrm{pr}_{\ker(A)}(\gamma), \gamma\rangle$$
$$= \mathrm{tr}_{\mathbb{C}}(\mathrm{pr}_{\ker(A)}) = \dim_{\mathbb{C}}(\ker(A))$$

Approximation in terms of spectral measures

- $\Gamma = \Gamma_1 > \Gamma_2 > \ldots$ residual chain.
- Let $a \in \mathbb{Z}[\Gamma]$.
- μ spectral measure of $r_a \colon \ell^2(\Gamma) \to \ell^2(\Gamma)$, i.e.

$$\int_{\mathbb{R}} f d\mu = \mathrm{tr}_\Gamma(f(a)).$$

- μ_i spectral measure of the reduction $r_{\bar{a}} \colon \mathbb{C}[\Gamma/\Gamma_i] \to \mathbb{C}[\Gamma/\Gamma_i]$.
  All measures are supported on some $[0, K]$.

$$\dim_\Gamma(\ker(r_a)) = \lim_{i\to\infty} \frac{\dim_{\mathbb{C}} \ker(\mathbb{C}[\Gamma/\Gamma_i] \xrightarrow{\bar{r}_a} \mathbb{C}[\Gamma/\Gamma_i])}{[\Gamma : \Gamma_i]}$$

$$\Updownarrow$$

$$\int_{\mathbb{R}} \chi_{\{0\}} d\mu = \mu(\{0\}) = \lim_{i\to\infty} \mu_i(\{0\}) = \lim_{i\to\infty} \int_{\mathbb{R}} \chi_{\{0\}} d\mu_i$$

Broad strategy

Spectrum **around** zero reveals something about the spectrum **at** zero.

Digression: Spectrum around zero

Chain complex in low degrees

Let Γ be a group with finite generating set $S = S^{-1}$. Let X be a classifying space whose 1-skeleton is the Cayley graph.

$$\underbrace{\hom_{\mathbb{Z}[\Gamma]}(C_0(X), \ell^2(\Gamma))}_{\ell^2(\Gamma)} \xrightarrow{d} \underbrace{\hom_{\mathbb{Z}[\Gamma]}(C_1(X), \ell^2(\Gamma))}_{\bigoplus_S \ell^2(\Gamma)} \to \cdots$$

starts the chain complex from which we compute $\beta_*^{(2)}(\Gamma)$.

Laplacian in degree 0 and its spectrum

$\Delta = d^* \circ d \colon \ell^2(\Gamma) \to \ell^2(\Gamma)$ is right multiplication with

$$2|S|\big(1 - \underbrace{\frac{1}{|S|}\sum_{s \in S} s}_{=:R}\big) \in \mathbb{C}[\Gamma].$$

$\mathrm{tr}_\Gamma(R^n)$ **return probability** of simple random walk on X after n steps. Its asymptotic is linked to the decay of the spectrum of Δ around zero.

An easy observation

For any $b = \sum b_\gamma \gamma \in \mathbb{C}[\Gamma]$ we have

$$\mathrm{tr}_\Gamma(b) = \mathrm{tr}_{\Gamma/\Gamma_i}(\bar{b}) \text{ for } i \geq i_0.$$

where i_0 is such that: $\gamma \in \Gamma \backslash \{e\},\ b_\gamma \neq 0 \Rightarrow \gamma \notin \Gamma_{i_0}$.

Weak convergence

$$\text{Apply to } b = a^n\text{:}\ \int_{\mathbb{R}} x^n d\mu(x) = \mathrm{tr}_\Gamma(a^n) = \lim_{i\to\infty} \mathrm{tr}_{\Gamma/\Gamma_i}(a^n) = \int_{\mathbb{R}} x^n d\mu_i(x)$$

Also true if $f(x) = x^n$ is replaced by a continuous function.

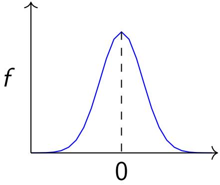
Caveat

Let $\nu_i = i \cdot \chi_{[0,1/i]} d\lambda$. Then

$$\nu_i \to \delta_0 \ \text{ weakly but } \ 0 = \nu_i(\{0\}) \not\to \delta_0(\{0\}) = 1.$$

Basic measure theory

$$\limsup_{i\to\infty} \mu_i(\{0\}) \leq \limsup_{i\to\infty} \int_{\mathbb{R}} f d\mu_i$$
$$= \lim_{i\to\infty} \int_{\mathbb{R}} f d\mu_i = \int_{\mathbb{R}} f d\mu \leq \mu(\{0\}) + \epsilon$$



Similarly for closed A and open U:

$$\limsup_{i\to\infty} \mu_i(A) \leq \mu(A) \text{ and } \liminf_{i\to\infty} \mu_i(U) \geq \mu(U)$$

Already proven: Kazhdan's inequality

Let X be a finite CW complex and $\pi_1(X) = \Gamma_0 > \Gamma_1 > \dots$ be a residual chain. Let $X_i \to X$ be the covering associated to Γ_i. Then

$$\limsup_{i\to\infty} \frac{b_n(X_i)}{[\Gamma : \Gamma_i]} \leq \beta_n^{(2)}(X)$$

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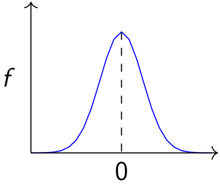

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Still to do

$$\liminf_{i\to\infty} \mu_i(\{0\}) \geq \mu(\{0\})$$

Integrality

- **Fix** i and let $n = [\Gamma : \Gamma_i]$. Let $0 = \lambda_1 = \ldots = \lambda_m < \lambda_{m+1} \leq \ldots \leq \lambda_n$ be the eigenvalues (with multiplicity) of $\bar{r}_a \colon \mathbb{C}[\Gamma/\Gamma_i] \to \mathbb{C}[\Gamma/\Gamma_i]$.
- Characteristic polynomial $p(z) = z^m q(z)$, $q \in \mathbb{Z}[z]$.
- $\lambda_{m+1} \cdots \lambda_n = q(0) \geq 1$.

Small eigenvalues

- Let $N(\epsilon)$ be the number of eigenvalues in $(0, \epsilon)$.
- $1 \leq \lambda_{m+1} \cdots \lambda_n \leq \epsilon^{N(\epsilon)} \|\bar{r}_a\|^n \leq \epsilon^{N(\epsilon)} \cdot \mathrm{const}^n$.
- $\mu_i((0,\epsilon)) = \frac{N(\epsilon)}{n} \leq \frac{\mathrm{const}}{|\log \epsilon|}$. Now **unfix** i.

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Conclusion of proof

$$\liminf_{i\to\infty} \mu_i(\{0\}) = \liminf_{i\to\infty} \big(\mu_i([0,\epsilon)) - \mu_i((0,\epsilon))\big) \geq \liminf_{i\to\infty} \underbrace{\mu_i([0,\epsilon))}_{=\mu_i((-\epsilon,\epsilon))} - \frac{\mathrm{const}}{|\log \epsilon|}$$

$$\geq \mu(\{0\}) - \frac{\mathrm{const}}{|\log \epsilon|}$$

Finally, let $\epsilon \to 0$!

Approximation theorem (Lück)

Let $\Gamma = \Gamma_0 > \Gamma_1 > \ldots$ be a residual chain. Let X be a finite free Γ-CW complex. Then

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Characteristic p

Let $\Gamma = \Gamma_0 > \Gamma_1 > \ldots$ be a residual chain. Let X be a finite free Γ-CW complex. What is

$$\lim_{i\to\infty} \frac{b_n(\Gamma_i \backslash X; \mathbb{F}_p)}{[\Gamma : \Gamma_i]} = ?$$

- Existence?
- Independence of (Γ_i)?
- $> \beta_n^{(2)}(X; \Gamma)$?

Need to find potential limit candidates, at least in specific situations!

Characteristic p

Let $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ be a residual chain. Let X be a finite free Γ-CW complex. What is

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Need to find potential limit candidates, at least in specific situations!

Results by Lackenby in degree 1

Let Γ be finitely presented and $b_1(\Gamma) > 0$. If the above limit is > 0 for a specific residual chain and some prime, then Γ is **large**.

Linnell's work on the Atiyah conjecture

His work is based on localization techniques.

$$\begin{array}{ccc} \mathbb{C}[\Gamma] & \hookrightarrow & L(\Gamma) \\ \downarrow & & \downarrow \\ \mathcal{D}(\Gamma) & \hookrightarrow & \mathcal{U}(\Gamma) \end{array}$$

- $\mathcal{U}(\Gamma)$ is the algebra of Γ-equivariant unbounded operators $\ell^2(\Gamma) \to \ell^2(\Gamma)$.
- $\mathcal{D}(\Gamma)$ is the division closure of $\mathbb{C}[\Gamma]$ inside $\mathcal{U}(\Gamma)$; serves as a localization of $\mathbb{C}[\Gamma]$.

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- $\mathcal{D}(\Gamma)$ is the division closure of $\mathbb{C}[\Gamma]$ inside $\mathcal{U}(\Gamma)$; serves as a localization of $\mathbb{C}[\Gamma]$.
- For torsionfree solvable groups $\mathcal{D}(\Gamma)$ is a division ring and

$$\beta_i^{(2)}(X;\Gamma) = \dim_{\mathcal{D}(\Gamma)} H^i(\hom_{\mathbb{Z}[\Gamma]}(C_*(X), \mathcal{D}(\Gamma)) \in \mathbb{N}.$$

- **Goal:** Characterize $\mathcal{D}(\Gamma)$ as an algebraic localization which can be done for $\mathbb{F}_p[\Gamma]$ as well.

Group rings of elementary amenable groups

Let Γ be a torsionfree elementary amenable group. Then $\mathbb{F}_p[\Gamma]$ has no zero divisors (Kropholler-Linnell-Moody, Linnell) and its Ore localization $Q(\mathbb{F}_p[\Gamma])$ is a division ring.

Approximation

Let Γ be a torsionfree elementary amenable group and (Γ_i) be a residual chain. Let X be a finite free Γ-CW complex. Then

$$\lim_{i\to\infty} \frac{b_n(\Gamma_i\backslash X;\mathbb{F}_p)}{[\Gamma:\Gamma_i]} = \dim_{Q(\mathbb{F}_p[\Gamma])}\Big(H_n\big(Q(\mathbb{F}_p[\Gamma])\otimes_{\mathbb{F}_p[\Gamma]} C_*(X)\big)\Big).$$

(Linnell-Lück-S.)

Algebraic description of ℓ^2-Betti numbers

Replace $\mathbb{F}_p$ by $\mathbb{C}$ above and one obtains an algebraic description of ℓ^2-Betti numbers in this case.

(Completed) group rings

Up to finite index, $\mathbb{F}_p[[\Gamma]] = \lim_{i\to\infty} \mathbb{F}_p[\Gamma/\Gamma_i]$ has no zero-divisors, and its Ore localization is a division ring.

Approximation

Let $\Gamma \hookrightarrow GL_n(\mathbb{Z}_p)$ be an embedding and $\Gamma_i = \ker(\Gamma \to GL_n(\mathbb{Z}/p^i)$. Let X be a finite free Γ-CW complex. Then

$$\lim_{i\to\infty} \frac{b_n(\Gamma_i\backslash X; \mathbb{F}_p)}{[\Gamma : \Gamma_i]} = \mathrm{rk}_{\mathbb{F}_p[[\Gamma]]}\Big(H_n\big(\mathbb{F}_p[[\Gamma]] \otimes_{\mathbb{F}_p[\Gamma]} C_*(X)\big)\Big) \in \mathbb{Q}.$$

(Calegari-Emerton; Bergeron-Linnell-Lück-S.)

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Open problem

Can one use this to prove the zero-divisor and Atiyah conjecture for torsionfree linear groups?

Residually torsionfree nilpotent groups

Orderable groups

Such groups possess a strict total ordering invariant under left and right translations.

Malcev-Neumann construction

Let k be a field. The ring of formal power series $k[[\Gamma]]$ with well-ordered support is a skew field containing $k[\Gamma]$.

Approximation

Let $\Gamma = \Gamma_0 > \Gamma_1 > \cdots$ be a normal chain such that $\bigcap_i \Gamma_i = \{1\}$ and each Γ/Γ_i is torsion-free nilpotent. Set $H_i = \Gamma_i \Gamma^{p^i}$.

$$\dim_{\mathbb{F}_p((\Gamma))}\big(H_n(\mathbb{F}_p((\Gamma)) \otimes_{\mathbb{F}_p[\Gamma]} C_*(X, \mathbb{F}_p))\big) = \lim_{i \to \infty} \frac{b_n(H_i \backslash X; \mathbb{F}_p)}{[\Gamma : H_i]}.$$

(Bergeron-Linnell-Lück-S.)

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ℓ^2-Betti numbers of locally compact groups

$\exists$ Theory of ℓ^2-Betti numbers for unimodular locally compact groups due to Davis-Dymara-Januszkiewicz-Okun and Petersen.

Structure theory

A locally compact group G modulo its amenable radical $R(G)$ is a product of a semisimple Lie group and a totally disconnected group. (**Hilbert's 5th problem**).

Focus on totally disconnected groups

$$\beta_n^{(2)}(G,\mu) = \begin{cases} 0 & \text{if } R(G) \text{ is not compact;} \\ \beta_n^{(2)}(G/R(G), \mathrm{pr}_* \mu) & \text{otherwise.} \end{cases}$$

- $\beta_n^{(2)}(G)$ for semisimple Lie group G can be studied by ℓ^2-Betti numbers of its lattices (Borel).
- Künneth formula reduces computations to totally disconnected groups.

von Neumann algebra $L(G)$ of G

G acts on $L^2(G,\mu)$ by translations from the left and the right. The analog of $\mathbb{C}[\Gamma] \hookrightarrow L(\Gamma)$ is

$$\lambda\colon C_0(G) \to \mathcal{B}(L^2(G,\mu))^G =: L(G)$$
$$\lambda(\phi)(f)(h) = \int_G \phi(g) f(g^{-1}h) d\mu(g).$$

Semifinite trace on $L(G)$ for totally disconnected G

The analog of $\mathrm{tr}_\Gamma|_{\mathbb{C}[\Gamma]}$ does not extend to all of $L(G)$.

$$\mathrm{tr}_G\colon C_0(G) \to \mathbb{C},\ \phi \mapsto \phi(e)$$

$e \in G$ has a neighborhood basis of compact-open subgroups. Define

$$\mathrm{tr}_{(G,\mu)}\colon L(G)_+ \to [0,\infty],\ \ T \mapsto \sup_{K <_{co} G} \langle T(\chi_K), \chi_K \rangle / \mu(K)^2$$

Note that $\mathrm{tr}_{(G,\mu)}(\lambda(\chi_K)) = 1$.

von Neumann dimension

For a G-invariant closed subspace $A \subset L^2(G)$,

$$\dim_{(G,\mu)}(A) := \mathrm{tr}_{(G,\mu)}(\mathrm{pr}_A) \in [0,\infty]$$

and similarly for $A \subset L^2(G)^d$. In general, $\dim_G(L^2(G)) = \infty$.

Projections from compact-open subgroups

Let $K < G$ be compact-open. The projection onto the subspace of left K-invariant functions ${}^K L^2(G,\mu) \subset L^2(G,\mu)$ is $\lambda(\frac{1}{\mu(K)}\chi_K)$.

$$\dim_{(G,\mu)}\big({}^K L^2(G,\mu)\big) = \frac{1}{\mu(K)}$$

Extension to arbitrary $L(G)$-modules

An extension of $\dim_{(G,\mu)}$ to arbitrary $L(G)$-modules in the spirit of Lück's dimension theory for finite von Neumann algebras is possible (Petersen).

G-CW-complexes

A proper smooth G-CW complex is a CW-complex X with a cellular G-action such that each cell has a compact-open stabilizer. As a G-module, the cellular chain complex looks like

$$C_n(X) \cong \bigoplus_{K \in \mathcal{F}_n} \mathbb{Z}[G/K].$$

A **geometric model** of G is a proper smooth contractible G-CW complex that has finitely many G-orbits of cells in each dimension.

E.g. Affine Bruhat-Tits buildings of reductive p-adic groups are such.

Cayley-Abels graph

Let $K < G$ be compact-open. Let $S \subset G$ be a bi-K-invariant compact generating set of G. The Cayley-Abels Graph is

- Vertices: cosets G/K
- Edges from gK to gsK.
- There are one equivariant 0-cell and $|K \backslash S/K|$-equivariant 1-cells.

ℓ^2-Betti numbers

$$\beta_n^{(2)}(G,\mu) = \dim_{(G,\mu)}\Big(\bar{H}_c^n\big(G, L^2(G)\big)\Big)$$

If G acts on a proper smooth contractible G-CW complex with finitely many G-orbits of cells in each dimension, then

$$\beta_n^{(2)}(G,\mu) = \dim_{(G,\mu)}\Big(\bar{H}^n\big(\hom_G(C_*(X), L^2(G))\big)\Big)$$

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Remark

If $K_1, \ldots, K_d$ are the stabilizers of the G-orbits of n-cells, then

$$\hom_G(C_n(X), L^2(G)) \cong {}^{K_1}L^2(G) \oplus \ldots \oplus {}^{K_d}L^2(G).$$

Thus,

$$\beta_n^{(2)}(G) \leq \frac{1}{\mu(K_1)} + \ldots + \frac{1}{\mu(K_d)}.$$

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ℓ^2-Betti numbers of a lattice $\Gamma < G$

$$\beta_n^{(2)}(\Gamma) = \mathrm{covol}(\Gamma)\beta_n^{(2)}(G,\mu) \quad \text{(Kyed-Petersen-Vaes).}$$

Example

Let $G = SL_3(\mathbb{Q}_p)$ and X be the 2-dim. Bruhat-Tits building of G.

- one equivariant 2-cell with stabilizer B, the Iwahori subgroup of G;
- three equivariant 1-cells corresponding to the edges of the fundamental chamber. The stabilizer of each splits into $p+1$ many cosets of B.

Normalizing $\mu(B) = 1$, we get $\beta_2^{(2)}(G) \geq 1 - 3/(p+1)$.

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Application to deficiency of lattices

For every lattice $\Gamma < G = SL_3(\mathbb{Q}_p)$, we have

$$\operatorname{def}(\Gamma) \leq 1 - \beta_2^{(2)}(\Gamma) = 1 - \beta_2^{(2)}(G)\operatorname{covol}(\Gamma) \leq 1 - \left(1 - \frac{3}{p+1}\right)\operatorname{covol}(\Gamma).$$

- Let X be the Cayley complex of a presentation. Then

$$\begin{aligned} g - r = 1 - \chi(X) &= 1 - \beta_0^{(2)}(\Gamma) + \beta_1^{(2)}(\Gamma) - \beta_2^{(2)}(\widetilde{X};\Gamma) \\ &= 1 + \beta_1^{(2)}(\Gamma) - \beta_2^{(2)}(\widetilde{X};\Gamma). \end{aligned}$$

- But $\beta_2^{(2)}(\widetilde{X};\Gamma) \geq \beta_2^{(2)}(\Gamma)$ and $\beta_1^{(2)}(\Gamma) = 0$ by property (T).

The space of subgroups

The set Sub_G of closed subgroups of G can be endowed with a topology (Chabauty topology) that makes it compact. $H_n \to H$ iff

- for $h \in H$ there is $h_n \in H_n$ with $h = \lim h_n$.
- for convergent (h_{n_k}) with have $\lim h_{n_k} \in H$.

Invariant random subgroups

A conjugation invariant Borel probability measure on Sub_G is called an invariant random subgroup (IRS). The set of IRS becomes a compact space with respect to weak convergence.

Lattices and normal subgroups as IRS

Let $\Gamma < G$ be a lattice. The pushforward of the Haar measure under $G/\Gamma \to \mathrm{Sub}_G$, $g\Gamma \to g\Gamma g^{-1}$, is the IRS ν_Γ associated to Γ. The point measure concentrated at a closed normal subgroup is an IRS.

Stuck-Zimmer theorem

Every non-atomic ergodic IRS in a connected simple Lie group of higher rank is of the form ν_Γ for a lattice Γ.

Levit: also true for simple algebraic groups over non-archimedean fields.

Margulis' normal subgroup theorem

Every normal subgroup of a lattice Γ in a higher rank simple Lie group is either finite or finite index in Γ.

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Stuck-Zimmer $\Rightarrow$ Margulis

Let $\Lambda \triangleleft \Gamma < G$.

- Consider pushforward ν of $G/\Gamma \to \mathrm{Sub}_G$, $g\Lambda \mapsto g\Lambda g^{-1}$.
- ν atomic? Then $\Lambda < Z(G)$ center.
- Otherwise Λ is a lattice by Stuck-Zimmer, thus $[\Gamma : \Lambda] < \infty$.

Automatic convergence (7s)

If (Γ_i) is a sequence of lattices in a higher rank simple Lie groups with $\mathrm{covol}(\Gamma_i) \to \infty$, then $\nu_{\Gamma_i} \to \delta_e$. (Also true in the p-adic case and in positive characteristic provided uniform discreteness by Gelander-Levit)

Uniform discreteness

A family of lattices is uniformly discrete, if there is a neighborhood of $e \in G$ that intersects every conjugate of a element in the family trivially.

Lattice approximation in Lie groups (7s)

Let G be a non-compact simple Lie group. If (Γ_i) is a uniformly discrete sequence of lattices whose IRS converge to δ_e, then

$$\beta_n^{(2)}(G, \mu) = \lim_{i \to \infty} \frac{b_n(\Gamma_i)}{\mathrm{covol}(\Gamma_i)}.$$

Lattice approximation in t.d. groups (Petersen-S.-Thom)

Assume that G totally disconnected has a geometric model. Let (Γ_i) be a sequence of lattices whose IRS converge to δ_e. Then

$$\beta_n^{(2)}(G,\mu) \leq \liminf_{i\to\infty} \frac{b_n(\Gamma_i)}{\operatorname{covol}(\Gamma_i)}.$$

If, in addition, (Γ_i) is uniformly discrete, then

$$\beta_n^{(2)}(G,\mu) = \lim_{i\to\infty} \frac{b_n(\Gamma_i)}{\operatorname{covol}(\Gamma_i)}.$$

Corollary

Let $\mathbf{G}$ be a simple algebraic group. Let $G = \mathbf{G}(\mathbb{Q}_p)$. If (Γ_i) is a sequence of lattices in G such that $\operatorname{covol}(\Gamma_i) \to \infty$, then

$$\beta_n^{(2)}(G,\mu) \leq \liminf_{i\to\infty} \frac{b_n(\Gamma_i)}{\operatorname{covol}(\Gamma_i)}.$$

Remark

In the discrete case the opposite inequality (Kazhdan's inequality) holds by general considerations.